# Class 27

ITERATIVE IMPROVEMENT

THE SIMPLEX METHOD

## Iterative Improvement

Algorithm design technique for solving optimization problems

- Start with a feasible solution
- Repeat the following step until no improvement can be found:
  - change the current feasible solution to a "nearby" feasible solution with a better value of the objective function
- Return the last feasible solution as optimal

Major difficulty: Local optimum vs. global optimum

Examples:

- simplex method
- Ford-Fulkerson algorithm for maximum flow problem
- maximum matching of graph vertices
- Gale-Shapley algorithm for the stable marriage problem

# Linear Programming

*Linear programming* (LP) problem is to optimize a linear function of several variables subject to linear constraints:

maximize/minimize: $c_1 x_1 + ... + c_n x_n$

subject to*:* $a_{i\,1}x_1 + ... + a_{in} x_n$ [≤/≥/=] $b_i$, $i = 1,...,m$

$x_1 \geq 0, \ldots, x_n \geq 0$

The function $z = c_1 x_1 + ... + c_n x_n$ is called the *objective function*;

constraints $x_1 \geq 0, \ldots, x_n \geq 0$ are called *non-negativity constraints*

Matrix/vector form:

$$\begin{array}{ll} \text{maximize} & \mathbf{c}^{\mathrm{T}}\mathbf{x} \\ \text{subject to} & A\mathbf{x} \leq \mathbf{b} \\ \text{and} & \mathbf{x} \geq \mathbf{0} \end{array}$$

# Example

Maximize: $3x + 5y$

subject to: $x + y \leq 4$

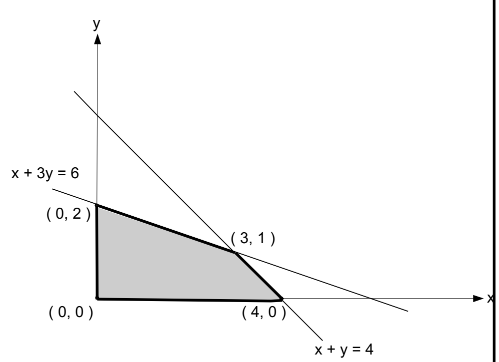
$x + 3y \leq 6$

$x \geq 0, \; y \geq 0$

The f*easible solution is any i*s any point (x, y) that satisfies all the constraints of the problem.

The f*easible region* of this problem is the set of all feasible points.

At right is a visualization of the feasible region.



# Geometric Solution

Task: Find an optimal solution.

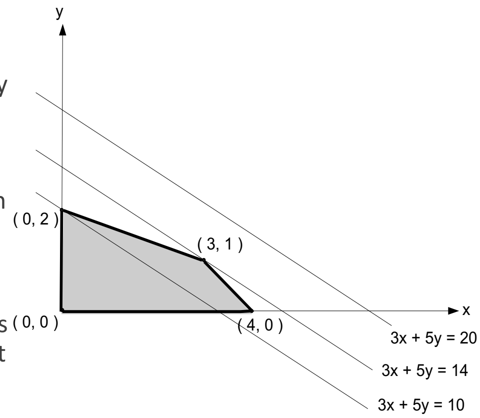
This is a point in the feasible region with the largest value of the objective function, i.e.: 3x + 5y

Optimal solution: 14,
when x = 3, y = 1

Consider: 3x + 5y = 20, no solution

Consider 3x + 5y = 10, many solutions

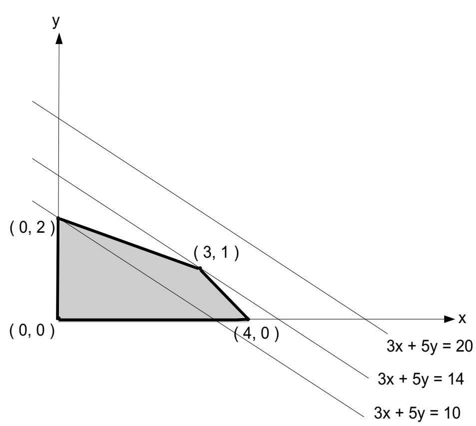
The lines defined by the equations above and shown on the right a called *level lines.*




# Geometric Solution

Objective: find largest parameter z for which the *level line* 3x + 5y = z has a common point with the feasible region.

What if the objective function were to be: 3x + 3y = z?



# 3 Possible Outcomes to LP Problem

*Non-unique*: the problem has a finite optimal solution, but it is not unique as we have seen on the prior slide.

*Infeasible*:

- there are no points satisfying all the constraints, i.e. the constraints are contradictory.
- Example: x + y <= 1 and x + y >= 2

# Extreme Point Theorem

Theorem: Any linear programming problem with a non-empty, bounded feasible region has an optimal solution. Moreover, an optimal solution can always be found at an extreme point of the problem's feasible region.

# Simplex Method

The classic method for solving LP problems.

One of the most important algorithms ever invented:
Top 10 Algorithms of the 20th Century

Based on the iterative improvement idea:

1. Start by identifying an extreme point of the feasible region
2. Then check whether one can get an improved value of the objective function by going to an adjacent extreme point.
3. If not, the current point is optimal : stop
4. Otherwise back to step (2)

# Simplex Method in Detail

First step: put LP problem in *standard form*

## Standard Form of an LP Problem

The standard from has the following requirements:

- It must be a **maximization** problem.
- All constraints (except the nonnegativity constraints) must be in the form of linear **equations** with nonnegative right-hand sides.
- All the variables must be required to be **nonnegative**

## Turning into Maximization Problem

A minimization problem can be converted into a maximization problem as follows:

-> Replace all coefficients with negative coefficients.

## Constraints in Form of Linear Equation

If a constraints is given as an inequality, it can be replaced by an equivalent equation by adding a **slack** variable.

For example: $x + y \leq 4$ ➡ $x + y + u = 4$ where $u \geq 0$

Note: a [≥] inequality can be turned into a [≤] via multiplication by (–1)

## Using Nonnegative Variables

All the variables must be required to be **nonnegative**

An unconstrained variable $x_j$ can be replaced by the difference between two new nonnegative variables:

-> substitute $x_j = x_j' - x_j''$, where $x_j', x_j'' \geq 0$

# Example of Standard Form

The general linear programming problem in standard form with $m$ constraints and $n$ unknowns ($n \geq m$) is

maximize: $c_1 x_1 + \ldots + c_n x_n$

subject to: $a_{i1}x_1 + \ldots + a_{in} x_n = b_i$, where $b_i >= 0$ for $i = 1,\ldots,m$,

$x_1 \geq 0, \ldots, x_n \geq 0$

maximize $3x + 5y$
subject to $x + y \leq 4$
$x + 3y \leq 6$
$x \geq 0, \quad y \geq 0$

➡

maximize $3x + 5y + 0u + 0v$
subject to $x + y + u = 4$
$x + 3y + v = 6$
$x \geq 0, \; y \geq 0, \; u \geq 0, \; v \geq 0$

# Basic Feasible Solutions

A *basic solution* to a system of $m$ linear equations in $n$ unknowns ($n \geq m$) is obtained by setting $n - m$ variables to 0 and solving the resulting system to get the values of the other $m$ variables

- The variables set to 0 are called *nonbasic*
- The variables obtained by solving the system are called *basic*
- A basic solution is called *feasible* if all its (basic) variables are nonnegative.

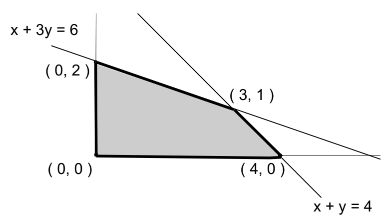
Example:

$x + y + u = 4$

$x + 3y + v = 6$

For (x, y, u, v), (0, 0, 4, 6) is a basic feasible solution

($x$, $y$ are nonbasic; $u$, $v$ are basic)




If we set $x$ and $u$ to zero, we obtain the basic solution(0, 4, 0, -6) which is not feasible

# Simplex Tableau for (0, 0, 4, 6)

Simplex method progresses through points.
A point can be represented by a simplex tableau.

maximize $z = 3x + 5y + 0u + 0v$

subject to $x + y + u = 4$

$x + 3y + v = 6$

$x \geq 0,\ y \geq 0,\ u \geq 0,\ v \geq 0$

basic variables → u, v

| | $x$ | $y$ | $u$ | $v$ | |
|---|---|---|---|---|---|
| $u$ | 1 | 1 | 1 | 0 | 4 |
| $v$ | 1 | 3 | 0 | 1 | 6 |
| objective row → | −3 | −5 | 0 | 0 | 0 ← value of $z$ at (0, 0, 4, 6) |

# Simplex Tableau

Objective row is used to check whether a solution is optimal.

This is the case when all entries in it are non-negative, except possibly that one in the last column.

(0, 0, 4, 6) is not optimal.

We can bump up x.

Notice x <= 4 per and x <= 6, per

Take the minimum, i.e. x = 4.

This leads to (4, 0, 0, 2)

Plugging this into the equation to be maximized, we get z = 12.

maximize $z = 3x + 5y + 0u + 0v$

subject to $x + y + u = 4$

$x + 3y + v = 6$

$x \geq 0,\ y \geq 0,\ u \geq 0,\ v \geq 0$

# Simplex Tableau

Alternatively, we can bump up y.

Per and y can be at most 2.

If we keep x = 0, then u = 2 and v = 0.

This leads to (0, 2, 2, 0)

Plugging this into the maximize equation, we get z = 10.

maximize $z = 3x + 5y + 0u + 0v$

subject to $x + y + u = 4$

$x + 3y + v = 6$

$x \geq 0, \; y \geq 0, \; u \geq 0, \; v \geq 0$

# Objective Row

Starting with the initial objective row, we select the most negative entry.

In our case, this is -5, i.e. y

Motivation: by improving on the worst data item, we hope we get the largest improvement.

Interestingly, in our example, this choice does not lead to the largest improvement, see prior two slides.

We'll go for it anyway!

| | $x$ | $y$ | $u$ | $v$ | |
|---|---|---|---|---|---|
| $u$ | 1 | 1 | 1 | 0 | 4 |
| $v$ | 1 | 3 | 0 | 1 | 6 |
| | −3 | −5 | 0 | 0 | 0 |

# Objective Row

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| | $x$ | $y$ | $u$ | $v$ | |
|---|---|---|---|---|---|
| $u$ | 1 | 1 | 1 | 0 | 4 |
| $v$ | 1 | 3 | 0 | 1 | 6 |
| | −3 | −5 | 0 | 0 | 0 |

# Simplex Example

1) Pick the column with the lowest (negative) value in the objective row, i.e. *y*. Mark that column with an up arrow.

2) Divide the last column entry of row *u* by the column entry for *y*. i.e. 4/1 = 4.

3) Divide the last column entry for row *v* by the column entry for *v*, i.e. 6/3 = 2

4) Select the lower one, i.e. row *v*.

5) Mark that row with a left arrow.

| | | $x$ | $y$ | $u$ | $v$ | |
|---|---|---|---|---|---|---|
| | $u$ | 1 | 1 | 1 | 0 | 4 |
| ← | $v$ | 1 | 3 | 0 | 1 | 6 |
| | | −3 | −5 | 0 | 0 | 0 |
| | | | ↑ | | | |

# Simplex Example

6) Divide the entries in the row with the left arrow by the value that is at the intersection of the up and left arrows, i.e. 3.

| | x | y | u | v | |
|---|---|---|---|---|---|
| u | 1 | 1 | 1 | 0 | 4 |
| ← v | 1 | 3 | 0 | 1 | 6 |
| | −3 | −5 | 0 | 0 | 0 |
| | | ↑ | | | |

| | x | y | u | v | |
|---|---|---|---|---|---|
| u | | | | | |
| y | $\frac{1}{3}$ | 1 | 0 | $\frac{1}{3}$ | 2 |
| | | | | | |

# Simplex Example

7) Replace the remaining rows as follows:

row = row – c * new_row_with_left_arrow, c is the entry at the intersection of that row with the up_arrow row.

For example, the <u, x> entry is calculated as follows: 1 – 1 * 1/3 = 2/3

| | x | y | u | v | |
|---|---|---|---|---|---|
| u | 1 | 1 | 1 | 0 | 4 |
| ← v | 1 | 3 | 0 | 1 | 6 |
| | −3 | −5 | 0 | 0 | 0 |
| | | ↑ | | | |

| | x | y | u | v | |
|---|---|---|---|---|---|
| u | $\frac{2}{3}$ | 0 | 1 | $-\frac{1}{3}$ | 2 |
| y | $\frac{1}{3}$ | 1 | 0 | $\frac{1}{3}$ | 2 |
| | | | | | |

# Simplex Example

7) Replace the remaining rows as follows:

row = row – c * new_row_with_left_arrow, c is the entry at the intersection of that row with the up_arrow row.

For example, the <objective_row, x> entry is calculated as follows:

(-3) – (-5) * 1/3 = - 4/3

| | $x$ | $y$ ↑ | $u$ | $v$ | |
|---|---|---|---|---|---|
| $u$ | 1 | 1 | 1 | 0 | 4 |
| ← $v$ | 1 | 3 | 0 | 1 | 6 |
| | −3 | −5 | 0 | 0 | 0 |

| | $x$ | $y$ | $u$ | $v$ | |
|---|---|---|---|---|---|
| $u$ | $\frac{2}{3}$ | 0 | 1 | $-\frac{1}{3}$ | 2 |
| $y$ | $\frac{1}{3}$ | 1 | 0 | $\frac{1}{3}$ | 2 |
| | $-\frac{4}{3}$ | 0 | 0 | $\frac{5}{3}$ | 10 |

# Simplex Example

8) replace the name of the row with the left-arrow with the name of the variable of the up arrow.

Notice that the basic solution for (x, y, u, v) is (0, 2, 2, 0) with a value of z = 10

| | $x$ | $y$ ↑ | $u$ | $v$ | |
|---|---|---|---|---|---|
| $u$ | 1 | 1 | 1 | 0 | 4 |
| ← $v$ | 1 | 3 | 0 | 1 | 6 |
| | −3 | −5 | 0 | 0 | 0 |

basic feasible sol.
(0, 0, 4, 6)

$z$ = 0

| | $x$ ↑ | $y$ | $u$ | $v$ | |
|---|---|---|---|---|---|
| ← $u$ | $\frac{2}{3}$ | 0 | 1 | $-\frac{1}{3}$ | 2 |
| $y$ | $\frac{1}{3}$ | 1 | 0 | $\frac{1}{3}$ | 2 |
| | $-\frac{4}{3}$ | 0 | 0 | $\frac{5}{3}$ | 10 |

basic feasible sol.
(0, 2, 2, 0)

$z$ = 10

# Simplex Example

Small group work time: Calculate the next tableaux.

| | $x$ | $y$ | $u$ | $v$ | |
|---|---|---|---|---|---|
| $u$ | 1 | 1 | 1 | 0 | 4 |
| ← $v$ | 1 | 3 | 0 | 1 | 6 |
| | −3 | −5 | 0 | 0 | 0 |
| | | ↑ | | | |

basic feasible sol.
(0, 0, 4, 6)

$z = 0$

| | $x$ | $y$ | $u$ | $v$ | |
|---|---|---|---|---|---|
| ← $u$ | $\frac{2}{3}$ | 0 | 1 | $-\frac{1}{3}$ | 2 |
| $y$ | $\frac{1}{3}$ | 1 | 0 | $\frac{1}{3}$ | 2 |
| | $-\frac{4}{3}$ | 0 | 0 | $\frac{5}{3}$ | 10 |
| | ↑ | | | | |

basic feasible sol.
(0, 2, 2, 0)

$z = 10$

| | $x$ | $y$ | $u$ | $v$ | |
|---|---|---|---|---|---|
| $x$ | 1 | 0 | $\frac{3}{2}$ | $-\frac{1}{2}$ | 3 |
| $y$ | 0 | 1 | $-\frac{1}{2}$ | $\frac{1}{2}$ | 1 |
| | 0 | 0 | 2 | 1 | 14 |

basic feasible sol.
(3, 1, 0, 0)

$z = 14$

# Simplex Algorithm Steps

Step 0 [Initialization] Present LP problem in standard form, set up initial tableau.

Step 1 [Optimality test] If all entries in the objective row are nonnegative, stop: the tableau represents an optimal solution.

Step 2 [Find entering variable] Select (the most) negative entry in the objective row. Mark its column to indicate the entering variable/pivot column.

Step 3 [Find departing variable] For each positive entry in the pivot column, calculate the θ-ratio by dividing that row's entry in the rightmost column by its entry in the pivot column. (If there are no positive entries in the pivot column, stop: the problem is unbounded.) Find the row with the smallest θ-ratio, mark this row to indicate the departing variable and the pivot row.

Step 4 [Form the next tableau] Scale pivot row so that pivot location becomes 1. Use pivot row to eliminate, so that rest of pivot column (including objective row) becomes 0's. Replace the label of the pivot row by the variable's name of the pivot column and go back to Step 1.

# Notes on the Simplex Method

- Finding an initial basic feasible solution may pose a problem
- Theoretical possibility of cycling
Typical number of iterations is between $m$ and $3m$, where $m$ is the number of equality constraints in the standard form
- Worst-case efficiency is exponential
- More recent *interior-point algorithms* such as *Karmarkar's algorithm* (1984) have polynomial worst-case efficiency and have performed competitively with the simplex method in empirical tests