# Day 16

GAUSSIAN ELIMINATION

LU (LOWER/UPPER) DECOMPOSITION

## Solving Linear Systems

Problem: solve linear system **Ax** = **b**, where **A** is *n* × *n*,…

Gaussian Elimination: Transforms to a system in upper triangular form **Ux** = **c**

From there, perform back-substitution.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned} \quad \Longrightarrow \quad \begin{aligned} a'_{11}x_1 + a'_{12}x_2 + \cdots + a'_{1n}x_n &= b'_1 \\ a'_{22}x_2 + \cdots + a'_{2n}x_n &= b'_2 \\ &\vdots \\ a'_{nn}x_n &= b'_n. \end{aligned}$$

# Example Transformation

*Linear System* — *Augmented Matrix*

$$\begin{cases} x+2y-4z=5 \\ 2x+y-6z=8 \\ 4x-y-12z=13 \end{cases} \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -4 & 5 \\ 2 & 1 & -6 & 8 \\ 4 & -1 & -12 & 13 \end{array}\right]$$

↓ Gaussian elimination

$$\begin{cases} x+2y-4z=5 \\ -3y+2z=-2 \\ -2z=-1 \end{cases} \Leftarrow \left[\begin{array}{ccc|c} 1 & 2 & -4 & 5 \\ 0 & -3 & 2 & -2 \\ 0 & 0 & -2 & -1 \end{array}\right]$$

*Equivalent System* — *Upper Triangular Form*

# Gaussian Elimination

Replace second equation with the difference between it and and the first equation, multiplied by $a_{21}/a_{11}$

Consider:

$$\begin{aligned} 2x_1 - x_2 + x_3 &= 1 \\ 4x_1 + x_2 - x_3 &= 5 \\ x_1 + x_2 + x_3 &= 0. \end{aligned}$$

Coefficients:

$$\begin{bmatrix} 2 & -1 & 1 & 1 \\ 4 & 1 & -1 & 5 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

TODO:

$$\begin{bmatrix} 2 & -1 & 1 & 1 \\ 4 & 1 & -1 & 5 \\ 1 & 1 & 1 & 0 \end{bmatrix} \begin{array}{l} \\ \text{row } 2 - \frac{4}{2}\text{ row } 1 \\ \text{row } 3 - \frac{1}{2}\text{ row } 1 \end{array}$$

# Gaussian Elimination

$$\begin{bmatrix} 2 & -1 & 1 & 1 \\ 4 & 1 & -1 & 5 \\ 1 & 1 & 1 & 0 \end{bmatrix} \begin{matrix} \\ \text{row } 2 - \frac{4}{2} \text{ row } 1 \\ \text{row } 3 - \frac{1}{2} \text{ row } 1 \end{matrix}$$

Outcome of the transformations:

$$\begin{bmatrix} 2 & -1 & 1 & 1 \\ 0 & 3 & -3 & 3 \\ 0 & \frac{3}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

# Gaussian Elimination

Now focus on the second column:

$$\begin{bmatrix} 2 & -1 & 1 & 1 \\ 0 & 3 & -3 & 3 \\ 0 & \frac{3}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

Subtract the third row from the second multiplying it by multiplied by $a_{32}/a_{22}$

$$\begin{bmatrix} 2 & -1 & 1 & 1 \\ 0 & 3 & -3 & 3 \\ 0 & \frac{3}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{matrix} \\ \\ \text{row } 3 - \frac{1}{2} \text{ row } 2 \end{matrix}$$

# Gaussian Elimination

Outcome:

$$\begin{bmatrix} 2 & -1 & 1 & 1 \\ 0 & 3 & -3 & 3 \\ 0 & 0 & 2 & -2 \end{bmatrix}$$

We can now obtain the solution by back substitution:
$x_3 = (-2)/2 = -1$
$x_2 = (3-(-3)x_3)/3) = 0$
$x_1 = (1 – x_3 –(-1)x_2)/2 = 1$

Complete problem 1 on worksheet.

# LU Decomposition

Consider our coefficients matrix from the prior example:

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 4 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

Consider the lower-triangular matrix, consisting of the multiples used in the transformation process:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix}$$

Consider the upper-triangular matrix, that is the result of the transformation:

$$U = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 3 & -3 \\ 0 & 0 & 2 \end{bmatrix}$$

# LU Decomposition

It turns out that A = L x U.

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix} \quad U = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 3 & -3 \\ 0 & 0 & 2 \end{bmatrix} \quad A = \begin{bmatrix} 2 & -1 & 1 \\ 4 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

Hence solving $Ax = b$ is equivalent to solving $LUx = b$

Denote $y = Ux$, substitute to obtain: $Ly = b$

First solve $Ly = b$, then $Ux = y$

# LU Decomposition

Solving $Ly = b$:

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 0 \end{bmatrix}$$

$y_1 = 1$

$y_2 = 5 – 2y_1 = 3$

$y_3 = 0 – y_1/2 – y_2/2 = -2$

# LU Decomposition

Solving *Ux = y:*

$$\begin{bmatrix} 2 & -1 & 1 \\ 0 & 3 & -3 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$$

$x_3$ = (-2)/2 = -1

$x_2$ = (3-(-3*$x_3$))/3 = 0

$x_1$ = (1 – (1* $x_3$) – ((-1) * $x_2$)/ 2 = 1

This is the same as calculated before.

# LU Decomposition - Recap

Gaussian Elimination reduces **A** to upper-triangular **U**

**L** contains coefficients in the reduction steps

Intuition: the decomposition **A** = **LU** “encodes” the steps of Gaussian Elimination

# Use of LU Decomposition

On the Top 10 Algorithms of the 20th Century

Knowing **LU** decomposition makes solving system **Ax** = **b** easy

**LUx** = **b**, equivalent to **L(Ux)** = **b**

- First solve **Ly** = **b** for **y** using forward-substitution
- Then solve **Ux** = **y** for **x** using backward-substitution
- Each step $\Theta(n^2)$

- So, one-time $\Theta(n^3)$ cost to find **LU**, and $\Theta(n^2)$ cost thereafter to solve any system **Ax** = **b** for different **b**