Session overview

- Hausdorff measure
- Definition of fractal and fractal dimension
Box-counting dimension

- Impose a grid of scale $s$.
- Count the number of cells, $N(s)$, that contain the shape.
- Repeat with increasingly larger $s$.
- Plot $(\log(N(s)))$ as a function of $\log(1/s)$ (a “log-log” plot, used to reveal exponential relationships).
- The box-counting dimension, $D_b$, is the slope of the best-fit line between these points.
- Details in PJS 4.2


Box-counting dimension

\[
m = \frac{\log 283 - \log 194}{\log 32 - \log 24} = \frac{2.45 - 2.29}{1.51 - 1.38} \approx 1.31
\]

Fractal appearance;
Notice dimension > 1,
similar to Koch curve
Hausdorff measure (PJS 4.4)

- Let $S \subseteq \mathbb{R}^n$ and let $B_i \subseteq \mathbb{R}^n$
- Denote the diameter of $B_i$ by $\text{diam}(B_i) = \sup \{ d(x,y) : x, y \in B_i \}$, the greatest distance across $B_i$
- Define $H_\varepsilon^h(S) = \inf \{ \sum_i [\text{diam}(B_i)]^h : \text{diam}(B_i) \leq \varepsilon \text{ and } \{B_i\} \text{ covers } S \}$
- Then, the Hausdorff $h$-dimensional measure of $S$ is $H^h(S) = \lim_{\varepsilon \to 0} H_\varepsilon^h(S)$
Example 1

- Let $S = \{ (1, 2), (2, 3), (-1, 2) \} \subseteq \mathbb{R}^2$
- Let $B_1 = \{ (x, y): (x-1)^2 + (y-2)^2 < (\varepsilon/2)^2 \}$
- Let $B_2 = \{ (x, y): (x-2)^2 + (y-3)^2 < (\varepsilon/2)^2 \}$
- Let $B_3 = \{ (x, y): (x+1)^2 + (y-2)^2 < (\varepsilon/2)^2 \}$
- Think of $\{ B_i \}$ as a set of disks of diameter $= \varepsilon$
- As long as $\varepsilon < 1$, none of the disks centered at the points in $S$ will overlap, and we require only 3 disks to cover $S$
- To compute $H^h(S)$ we have $0^h + 0^h + 0^h = 3$ (if $h = 0$) or $0$ (if $h > 0$) (measure depends on dimension)
Example 2

- Let $S = \{ [0, 2] \times [0, 3] \} \subseteq \mathbb{R}^2$
- Let $B_i$ be a set of squares, each $1/n$ on a side, that covers $S$
- The diameter of each square is its diagonal, $(\sqrt{2})/n$
- $H^h(S)$ is computed by:

$$
\lim_{n \to \infty} \sum_{i=1}^{6n^2} \left( \frac{\sqrt{2}}{n} \right)^h = \lim_{n \to \infty} 6n^2 \left( \frac{\sqrt{2}}{n} \right)^h = 6(\sqrt{2})^h \lim_{n \to \infty} n^{2-h} = \begin{cases} 
\infty & h < 2 \\
6(\sqrt{2})^h & h = 2 \\
0 & h > 2
\end{cases}
$$
Remark

- $H^h(S)$ is always decreasing
- It's graph typically looks like

$$H^h(S)$$

- The Hausdorff dimension, $D_H$, is the critical value of $h$ where the dimension jumps from $\infty$ to 0:
  $$D_H(S) = \inf\{ h \mid H^h(S)=0 \} = \sup\{ h \mid H^h(S)=\infty \}$$
Example 3

- Let \( S \) = the middle-thirds Cantor set
- At each stage, we require \( \{ B_i \} \) to contain \( 2^n \) elements, each with diameter \( (\frac{1}{3})^n \) to cover the Cantor set without overdoing it
- Let \( \varepsilon = (\frac{1}{3})^n \) and let \( \varepsilon \rightarrow 0 \) by letting \( n \rightarrow \infty \)

\[
H^h_\varepsilon(S) = \sum_{i=1}^{2^n} \left[ \left( \frac{1}{3} \right)^n \right]^h = 2^n \left( \frac{1}{3^n} \right)^h
\]

\[
\therefore H^h(S) = \lim_{n \rightarrow \infty} 2^n \left( \frac{1}{3^n} \right)^h = \lim_{n \rightarrow \infty} \frac{2^n}{3^{nh}} = \lim_{n \rightarrow \infty} \left( \frac{2}{3^h} \right)^n
\]
Example 3 (cont.)

- \( H^0(S) = \lim_{n \to \infty} 2^n = \infty \)
- \( H^1(S) = \lim_{n \to \infty} (2/3)^n = 0 \)

Note:
- if \( 3^h > 2 \), then \( H^h(S) = 0 \)
- if \( 3^h < 2 \), then \( H^h(S) = \infty \)
- if \( 3^h = 2 \), then \( H^h(S) = 1 \)
- \( 3^h = 2 \Rightarrow h = \log 2 / \log 3 = 0.631 \)

To summarize, the Cantor set has:

\[
\begin{align*}
\text{Hausdorff 0-dimensional measure} &\quad \infty \\
\text{Hausdorff 1-dimensional measure} &\quad 0 \\
\text{Hausdorff 0.631-dimensional measure} &\quad 1 \\
\end{align*}
\]

- Summary: Hausdorff-dimension 0.631
- Self-similarity dimension 0.631
- Topological dimension 0
- Lebesgue measure 0
The Koch snowflake paradox

- The Koch snowflake fits inside a circle even though it has infinite length.
- This is not really a paradox if we think about length in the right way.
- Length is usually thought of as $L^1$ length (or measure), which is equivalent to $H^1$ measure, infinity in this case.
- Compute the Koch snowflake’s Hausdorff dimension $D_H$ (on quiz).
Equivalence of Hausdorff and self-similarity dimensions

- The Hausdorff dimension is equivalent to the self-similarity dimension for the case of self-similar fractals
Definition of fractal and fractal dimension

- Mandelbrot coined the term *fractal* in 1977
- A set $X \subseteq \mathbb{R}^n$ is said to be a *fractal* if its Hausdorff dimension, $D_H$, strictly exceeds its topological dimension, $D_T$
- The number $D_H$ is the *fractal dimension* of the set
Homework #1

- On the course web site
- Begin work on it in the remaining time we have in class