Disco II - Quiz 9 - Answers

Name:_____ Box #_____

1. Generating Functions

1. Find the coefficient of x^{10} in $\frac{1+x}{1-2x}$.

$$2^9 + 2^{10} = 1536$$

2. What are the generating functions of the following sequences.

   1. $0, 0, 0, 0, 1, -1, 1, -1, 1, -1, \ldots$

   $$\frac{x^4}{1+x}$$

   2. $2, 5, 13, 35, \ldots, 1, 2^n + 3^n, \ldots$ (general term in the n'th slot)

   $$\frac{1}{1-2x} + \frac{1}{1-3x} = \frac{2-5x}{(1-2x)(1-3x)}$$

   3. $0, 1, 3, 7, 15, \ldots, 1 + 2 + 4 + \cdots + 2^{n-1}, \ldots$ (general term in the n'th slot) or

   $0, 1, 3, 7, 15, \ldots, 2^n - 1, \ldots$ (general term in the n'th slot)

$$\begin{aligned} \frac{1}{1-x}\frac{x}{1-2x} &= \frac{x}{(1-2x)(1-x)} \text{ convolution solution} \\ \frac{1}{1-2x} - \frac{1}{1-x} &= \frac{x}{(1-2x)(1-x)} \text{ (difference of sequences solution)} \end{aligned}$$

3. Find the general term of sequence whose generating function is: $\frac{x}{(1-x)^2(1+2x)}$

$$\begin{aligned} \frac{x}{(1-x)^2(1+2x)} &= \frac{A}{(1-x)} + \frac{B}{(1-x)^2} + \frac{C}{1+2x} \\ x &= A(1-x)(1+2x) + B(1+2x) + C(1-x)^2 \\ x &= (-2A+C)\,x^2 + (A+2B-2C)\,x + A + B + C \end{aligned}$$

$$\begin{aligned} -2A + C &= 0, \ -2C + 2B + A = 1, \ A + B + C = 0 \\ A &= -1/9, B = 1/3, C = -2/9 \end{aligned}$$

$$\begin{aligned} \frac{x}{(1-x)^2(1+2x)} &= \frac{1}{9}\left(\frac{-1}{(1-x)} + \frac{3}{(1-x)^2} + \frac{-2}{1+2x}\right) \\ &= \frac{1}{9}\left(\sum_{k=0}^{\infty} -1x^k + \sum_{k=0}^{\infty} 3(k+1)x^k + \sum_{k=0}^{\infty} -2\cdot(-2)^k x^k\right) \\ &= \frac{1}{9}\left(\sum_{k=0}^{\infty}(-1+3(k+1)-2\cdot(-2)^k)x^k\right) \\ &= \sum_{k=0}^{\infty}\frac{2+3k+(-2)^{1+k}}{9}x^k \end{aligned}$$

$$\text{general term} = \frac{2+3k+(-2)^{1+k}}{9}$$