Disco II - Quiz 10

Name:____________________ Box #____________________

1. Using Generating Functions

1. Find the generating function and indicate what coefficient is required.

   a the number of solutions to $n = n_1 + n_2 + n_3 + n_4$, n_1 and n_3 odd n_2, and n_4 even, non negative integers.
   Find the coefficient of x^n in $f(x)$ where $f(x)$ is given by:

$$
\begin{aligned}
f(x) &= (x + x^3 + x^5 + \cdots)^2 (1 + x^2 + x^4 + \cdots)^2 \\
&= \left(\frac{1}{2}\left(\frac{1}{1-x} - \frac{1}{1+x}\right)\right)^2 \left(\frac{1}{2}\left(\frac{1}{1-x} + \frac{1}{1+x}\right)\right)^2 \\
&= \left(\frac{1}{2}\left(\frac{2x}{(1-x)(1+x)}\right)\right)^2 \left(\frac{1}{2}\left(\frac{2}{(1-x)(1+x)}\right)\right)^2 \\
&= \frac{x^2}{(1-x)^4 (1+x)^4} \\
&= \frac{x^2}{(1-x^2)^4}.
\end{aligned}
$$

   find the coefficient of x^n

   b the number of solutions to $n = n_1 + n_2 + n_3 + n_4, n_i \leq 3i$, all n_i nonnegative.

$$
\begin{aligned}
g(x) &= (1 + x + x^2 + x^3)(1 + x + x^2 + x^3 + x^4 + x^5 + x^6) \cdot \\
&\quad (1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9) \cdot \\
&\quad (1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + x^{10} + x^{11} + x^{12}) \\
&= \frac{(1-x^4)(1-x^7)(1-x^{10})(1-x^{13})}{(1-x)^4}
\end{aligned}
$$

2. An icosahedral die has the number 1 to 10 written on the faces with the following frequencies.

1	2	3	4	5	6	7	8	9	10
1	1	2	3	3	3	3	2	1	1

a Write out the generating function for the number of ways one can get integer n in a single toss.
The number of ways is the coefficient of n in $h(x)$

$$h(x) = x + x^2 + 2x^3 + 3x^4 + 3x^5 + 3x^6 + 3x^7 + 2x^8 + x^9 + x^{10}$$

b Now the number of way the die sum is n for two (independent) tosses.
The number of ways is the coefficient of n in

$$(h(x))^2 = (x + x^2 + 2x^3 + 3x^4 + 3x^5 + 3x^6 + 3x^7 + 2x^8 + x^9 + x^{10})^2$$

The following is not a required part of the solution

$$\begin{aligned}(h(x))^2 \quad = \quad & x^2 + 2x^3 + 5x^4 + 10x^5 + 16x^6 + 24x^7 + 33x^8 + \\ & 40x^9 + 45x^{10} + 48x^{11} + 45x^{12} + 40x^{13} + 33x^{14} + 24x^{15} + \\ & 16x^{16} + 10x^{17} + 5x^{18} + 2x^{19} + x^{20}\end{aligned}$$

c The same as 2 except there are k tosses.
The number of ways is the coefficient of n in

$$(h(x))^k = (x + x^2 + 2x^3 + 3x^4 + 3x^5 + 3x^6 + 3x^7 + 2x^8 + x^9 + x^{10})^k$$

The following is not a required part of the solution but illustrates how fast the coefficients grow:

$$\begin{aligned}(h(x))^3 \quad = \quad & x^3 + 3x^4 + 9x^5 + 22x^6 + 45x^7 + 84x^8 + 143x^9 + 222x^{10} + \\ & 321x^{11} + 435x^{12} + 549x^{13} + 654x^{14} + 735x^{15} + 777x^{16} + 777x^{17} + \\ & 735x^{18} + 654x^{19} + 549x^{20} + 435x^{21} + 321x^{22} + 222x^{23} + 143x^{24} + \\ & 84x^{25} + 45x^{26} + 22x^{27} + 9x^{28} + 3x^{29} + x^{30}\end{aligned}$$