Applied Math I - Final Exam

Professor Broughton February 22, 1998

Name:_____ Box #_____

Instructions

- Answer all the questions directly on the exam.
- Show all the necessary work and write your answers out neatly in English sentences. Use mathematical notation to express your answers, not Maple notation
- It is not necessary to use your computer to answer all of the questions but you can use it to obtain graphs, evaluate functions, solve equations, etc. If you use Maple/Matlab be sure to say so by some sentence such as: Using Maple/Matlab the above integral equals ....
- Recall that you may use notes that you can …t on one standard sheet of paper. On your computer you may start o¤ with one blank Maple worksheet only. The same rules apply if you use Matlab
- All questions are worth 20 points.

Question	Points
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
Total	

1.a Find the equilibrium displacements of the spring system pictured below.

1.b Find the general solution to the network system pictured below.

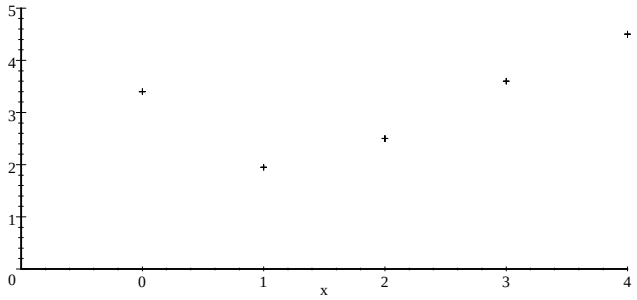
2.a Solve the following system of equations by hand, showing all the steps.

$$\begin{aligned} 2x + 1y + 3z + w &= 5 \\ 2x + 2y + 3z - 2w &= 5 \\ 10x + 8y + 15z - 4w &= 25 \end{aligned}$$

2.b The solution set is a 2D plane in four dimensional space. Where does this plane intersect the xw-plane?

3.a Consider the following data in table and graph form:

obs#	1	2	3	4	5
x_i	0	1	2	3	4
y_i	3:4	1:95	2:5	3:6	4:5



Find the quadratic …tting polynomial for this data, and plot the curve.

3.b Let $B = \begin{bmatrix} 3:4 \\ 1:95 \\ 2:5 \\ 3:6 \\ 4:5 \end{bmatrix}$ be the vector of observations, AX the vector of predictions obtained from the quadratic model and $E = B - AX$ the vector of errors, so that $B = AX + E$: Verify that "Pythagorean theorem" holds.

$$\|B\|^2 = \|AX\|^2 + \|E\|^2 :$$

4. Your company has produced 10,000 of a certain industrial controller board. This controller board has two components A and B and the board can be in one of four states:

State	A	B	Proportion
WW	working	working	x_n
WF	working	failed	y_n
FW	failed	working	z_n
FF	failed	failed	w_n

:

The board is considered functional if at least one of the components is working. Let x_n; y_n; z_n; and w_n represent the proportion of the controllers that are in a given state after n months as in the table above. Lab tests are predicting that during the month n some of the controller boards will fail

and/or be repaired according to the following formulas:

$$
\begin{aligned}
x_{n+1} &= 0{:}999x_n + 0{:}5y_n + 0{:}5z_n \\
y_{n+1} &= 0{:}0005x_n + 0{:}49y_n \\
z_{n+1} &= 0{:}0004x_n + 0{:}49z_n \\
w_{n+1} &= 0{:}0001x_n + 0{:}01y_n + 0{:}01z_n + w_n
\end{aligned}
$$

All 10,000 of the controllers are shipped at once in the WW state.

4.a Let $X_n = \begin{bmatrix} x_n \\ y_n \\ z_n \\ w_n \end{bmatrix}$: Then $X_{n+1} = AX_n$ for some A: Find A: What is X_0?

4.b You have bet all your future income with the company that 99% of the controller boards will be functional for at least 5 years. Are you in trouble? Suggestion: in previous work with this type of problem we have noted that $X_1 = AX_0$; $X_2 = A^2X_0$; etc.

5. The matrix A is given by:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 2 & 3 \\ 1 & 4 & 5 & 7 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

5.a Find the reduced row echelon form of A; and from that a basis of the null space of A:

5.b Observe that $\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ is a particular solution to AX = B where B = $\begin{bmatrix} 4 \\ 2 \\ 6 \\ 2 \end{bmatrix}$:
Using this particular solution and your answer in 5.a. Find the general solution to AX = B:

5.c Give a non-redundant system of additional constraints that will result in a unique solution.

6.a Solve the following non-linear D.E. by separation of variables, showing all steps.

$$x'(t) = x(t)(3 - x(t)); \; x(0) = 4:$$

6.b Make a direction …eld plot of the D.E. on the axes below and show the

solution to the D.E. above

:

6.c What values of the initial condition lead to a …nite limit as t ! 1 :

7.a An RC circuit as discussed in class has R = 30000 Ohms and C = 1 micro farad. The corresponding equation is

$$RCv'(t) + v(t) = b(t);\ v(0) = v_0$$

where v(t) is the voltage across the capacitor and b(t) is the input voltage. Using the convolution integral, write out the solution to the D.E. using the free response + zero initial condition forced response (homogeneous + particular) format. Use your general solution to solve the next two parts of this question.

7.b Now suppose that the circuit is initially dead but subjected to a unit impulse of one second duration starting at t = 3. What is the maximum of the response, and how long does it take to die away.

7.c Same as above, but now the input signal is cos(! t). What is the amplitude of the response after the transient has died away?

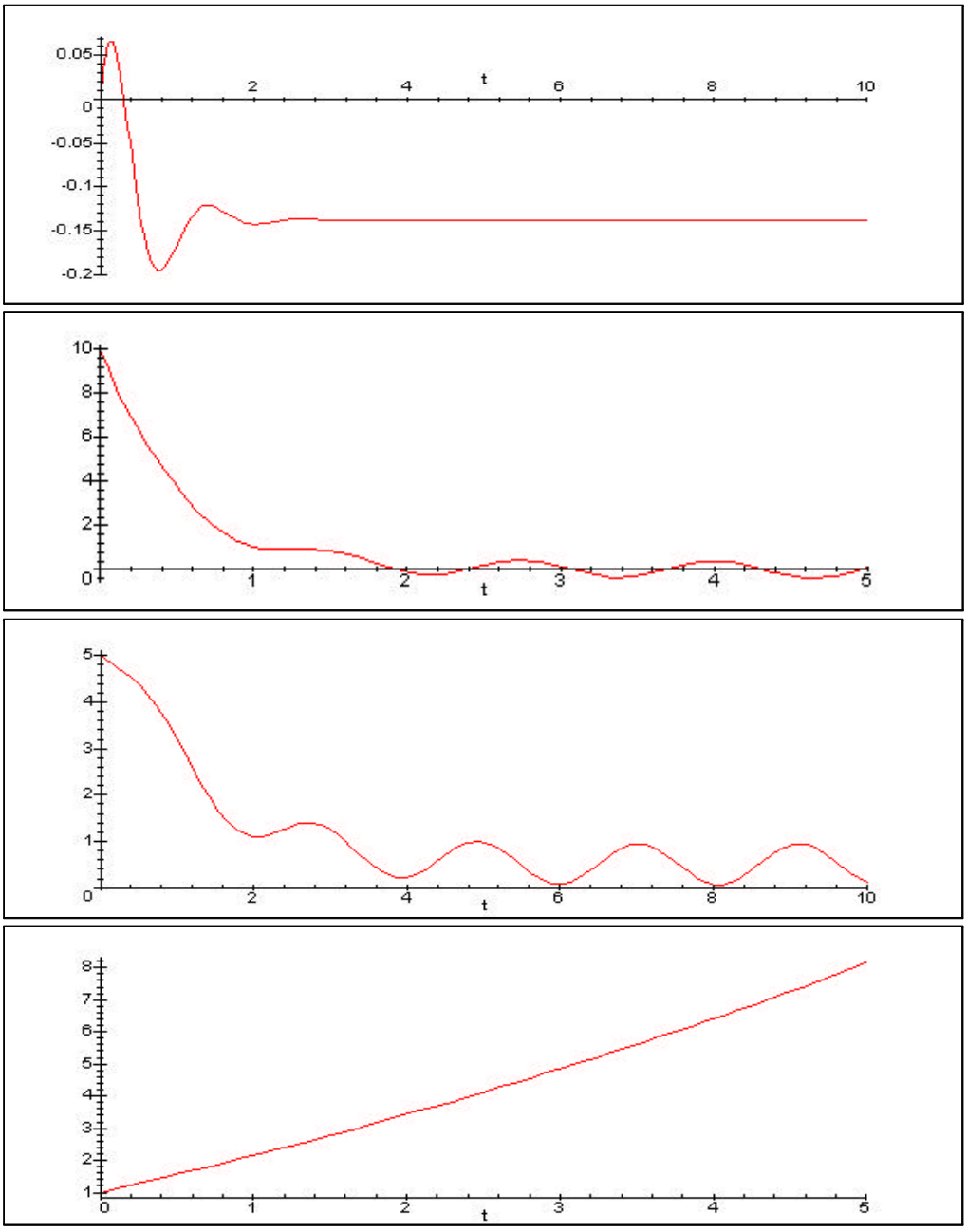
8. Find solution of the following equation using the method of undetermined coe¢cients, showing all the steps. Use may use Maple to check your solution.

$$z''(t) + 4z'(t) + 8z(t) = 3\cos(5t);\ z(0) = 1;\ z'(0) = \text{¡}1$$

9. Below we have four di¤erential equations. Fill in the table below the best you can. Write a one or two sentence justi…cation for each line of the table.

$$
\begin{aligned}
0{:}5y'(t) + y(t) &= \cos(at);\ y(0) = 10; \\
y'(t) - 0{:}1y(t) &= L;\ y(0) = 1; \\
y''(t) + 4y'(t) + 29y(t) &= L;\ y(0) = 5;\ y'(0) = -1 \\
y''(t) + 3y'(t) + 2y(t) &= a + \cos(bt);\ y(0) = 0;\ y'(0) = 1
\end{aligned}
$$

Case	Characteristic Roots	Stability	Time constant	Graph
1				
2				
3				
4				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-0.15
-0.2
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10 A salt tank has a capacity of 500 gals and has 25 lbs of salt. An inlet pipe is pumping ‡uid into the tank at 10 gals per minute at the top of the tank. Brine ‡ows out tank at the same rate. The originally scheduled plan was to pump fresh water into the tank until the concentration was down to one half the initial concentration. However, an error was made and the initially the inlet pipe was connected to a large brine pond with a salt concentration of 0.15 lbs/gal. The crew then went o¤ for a 45 minute co¤ee break. After co¤ee break was over, the crew discovered the mistake and hooked the up the inlet to the fresh water pond. They were unable to increase the rate of ‡ow in and out of the tank. Here are the questions:

- Did they initially have time to go on co¤ee break? (assume all shuto¤s are manual)
- How much are they behind schedule.

  Suggestion: model both scenarios with Heaviside functions and solve each using Maple. Write down the solutions and sketch the graphs. Estimate times from the graphs.