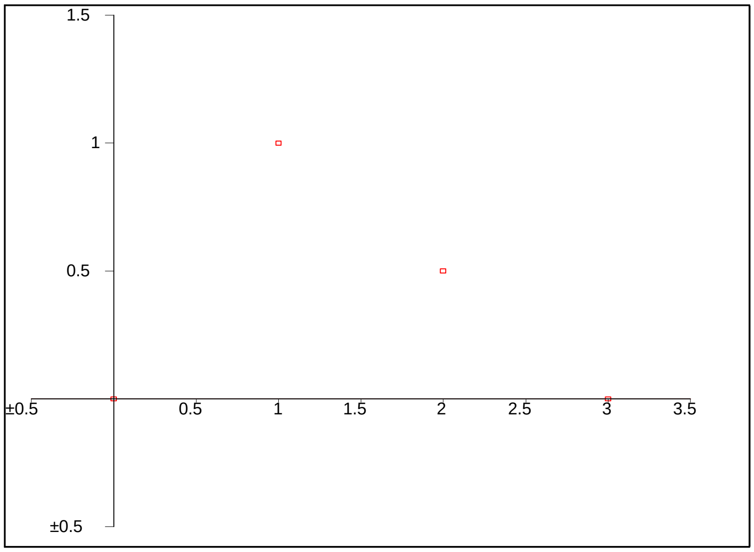
Applied Mathematics I - Spline example

Let us put a cubic spline through the points (0; 0), (1; 1) (2; 0:5) and (3; 0).



We shall assume that the spline meets the x ¡ axis at 45$^{\pm}$ at both ends The spline will be de…ned by three curves.

$$
\begin{aligned}
f_1(x) &= a_1x^3 + b_1x^2 + c_1x + d_1; 0 < x < 1;\\
f_2(x) &= a_2x^3 + b_2x^2 + c_2x + d_2; 1 < x < 2;\\
f_3(x) &= a_3x^3 + b_3x^2 + c_3x + d_3; 2 < x < 3:
\end{aligned}
$$

At the point (0; 0) we have these constraints:

$$
\begin{aligned}
f_1(0) &= 0;\\
f_1^{0}(0) &= 1;
\end{aligned}
$$

At (1; 1) we get

$$
\begin{aligned}
f_1(1) &= 1;\\
f_2(1) &= 1;\\
f_1^{0}(1) &= f_2^{0}(1);\\
f_1^{00}(1) &= f_2^{00}(1):
\end{aligned}
$$

Similarly at $(2, .5)$ and $(3, 0)$ we get:

$$\begin{aligned} f_2(2) &= 0.5, \\ f_3(2) &= 0.5, \\ f_2'(2) &= f_3'(2), \\ f_2''(2) &= f_3''(2), \end{aligned}$$

and

$$\begin{aligned} f_3(3) &= 0, \\ f_3'(3) &= -1, \end{aligned}$$

This yields the following 12 equations in 12 unknowns.

$$\begin{aligned}
0 \cdot a_1 + 0 \cdot b_1 + 0 \cdot c_1 + d_1 &= 0 \\
3 \cdot 0 \cdot a_1 + 2 \cdot 0 \cdot b_1 + c_1 &= 1 \\
1 \cdot a_1 + 1 \cdot b_1 + 1 \cdot c_1 + d_1 &= 1 \\
1 \cdot a_2 + 1 \cdot b_2 + 1 \cdot c_2 + d_2 &= 1 \\
3 \cdot 1 \cdot a_1 + 2 \cdot 1 \cdot b_1 + c_1 &= 3 \cdot 1 \cdot a_2 + 2 \cdot 1 \cdot b_2 + c_2 \\
6 \cdot 1 \cdot a_1 + 2 \cdot 1 \cdot b_1 &= 6 \cdot 1 \cdot a_2 + 2 \cdot 1 \cdot b_2 \\
8 \cdot a_1 + 4 \cdot b_1 + 2 \cdot c_1 + d_1 &= .5 \\
8 \cdot a_2 + 4 \cdot b_2 + 2 \cdot c_2 + d_2 &= .5 \\
3 \cdot 4 \cdot a_2 + 2 \cdot 3 \cdot b_2 + c_2 &= 3 \cdot 4 \cdot a_3 + 2 \cdot 2 \cdot b_3 + c_3 \\
27 \cdot a_3 + 9 \cdot b_3 + 3 \cdot c_3 + d_3 &= 0 \\
3 \cdot 9 \cdot a_3 + 2 \cdot 3 \cdot b_3 + c_3 &= -1
\end{aligned}$$

These equations have the form below where all the unmarked entries are zero.

$$
\begin{bmatrix}
0 & 0 & 0 & 1 & & & & & & & & \\
0 & 0 & 1 & 0 & & & & & & & & \\
1 & 1 & 1 & 1 & & & & & & & & \\
 & & & & 1 & 1 & 1 & 1 & & & & \\
3 & 2 & 1 & 0 & -3 & -2 & -1 & 0 & & & & \\
6 & 2 & 0 & 0 & -6 & -2 & 0 & 0 & & & & \\
 & & & & 8 & 4 & 2 & 1 & & & & \\
 & & & & & & & & 8 & 4 & 2 & 1 \\
 & & & & 12 & 4 & 1 & 0 & -12 & -4 & -1 & 0 \\
 & & & & 12 & 2 & 0 & 0 & -12 & -2 & 0 & 0 \\
 & & & & & & & & 27 & 9 & 3 & 1 \\
 & & & & & & & & 27 & 6 & 1 & 0
\end{bmatrix}
\begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \\ a_2 \\ b_2 \\ c_2 \\ d_2 \\ a_3 \\ b_3 \\ c_3 \\ d_3 \end{bmatrix}
=
\begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ .5 \\ .5 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}
$$

Rearranging the equations might make a better structure for the matrix. For example another arrangement is:

$$
\begin{bmatrix}
0 & 0 & 0 & 1 & & & & & & & & \\
0 & 0 & 1 & 0 & & & & & & & & \\
1 & 1 & 1 & 1 & & & & & & & & \\
 & & & & 1 & 1 & 1 & 1 & & & & \\
 & & & & 8 & 4 & 2 & 1 & & & & \\
 & & & & & & & & 8 & 4 & 2 & 1 \\
 & & & & & & & & 27 & 9 & 3 & 1 \\
 & & & & & & & & 27 & 6 & 1 & 0 \\
3 & 2 & 1 & 0 & -3 & -2 & -1 & 0 & & & & \\
 & & & & 12 & 4 & 1 & 0 & -12 & -4 & -1 & 0 \\
6 & 2 & 0 & 0 & -6 & -2 & 0 & 0 & & & & \\
 & & & & 12 & 2 & 0 & 0 & -12 & -2 & 0 & 0
\end{bmatrix}
\begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \\ a_2 \\ b_2 \\ c_2 \\ d_2 \\ a_3 \\ b_3 \\ c_3 \\ d_3 \end{bmatrix}
=
\begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ .5 \\ .5 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

The solution is:

$$
\begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \\ a_2 \\ b_2 \\ c_2 \\ d_2 \\ a_3 \\ b_3 \\ c_3 \\ d_3 \end{bmatrix} = \begin{bmatrix} \text{¡ ::73333} \\ \text{:73333} \\ 1 \\ 0 \\ \text{:70000} \\ \text{¡ 3:5666} \\ \text{5:3000} \\ \text{¡ 1:43333} \\ \text{¡ :56667} \\ \text{4:03333} \\ \text{¡ 9:90000} \\ \text{8:70000} \end{bmatrix}
$$

The graph of the spline is: