MA 112 - Calculus II
Worksheet # 5
Professor Broughton

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Due: Wednrsday Jan 15.

1. Compute the following integrals:

- $\int \frac{3x+2}{x^2-5x+6} dx$

$$x^2 - 5x + 6 = (x-2)(x-3).$$

Write
$$\frac{3x+2}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3}.$$
Then (using root substitution method)

$$\begin{aligned} 3x + 2 &= A(x-3) + B(x-2) \\ 3 \cdot 3 + 2 &= A \cdot 0 + B(3-2),\ B = 11 \\ 3 \cdot 2 + 2 &= A(2-3)\ + B \cdot 0,\ A = -8 \end{aligned}$$

$$\begin{aligned} \int \frac{3x+2}{x^2-5x+6} dx &= -8 \int \frac{dx}{x-2} + 11 \int \frac{dx}{x-3} \\ &= -8 \ln|x-2| + 11 \ln|x-3| + C \end{aligned}$$

- $\int_4^5 \frac{3x^4+2}{x^2-5x+6} dx$

Using long division we get:

$$\begin{aligned} 3x^4 + 2 &= (x^2 - 5x + 6)(3x^2 + 15x + 57) + 195x - 340 \\ \frac{3x^4+2}{x^2-5x+6} &= 3x^2 + 15x + 57 + \frac{195x - 340}{x^2 - 5x + 6} \end{aligned}$$

$$\frac{195x - 340}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3}$$
$$195x - 340 = A(x-3) + B(x-2)$$
$$195x - 340 = (A+B)x - 3A - 2B$$

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$$A + B = 195,\ 3A + 2B = 340$$
$$B = 195 - A,\ 3A + 2(195 - A) = 340$$
$$A = -50,\ B = 245$$

$$\int_4^5 \frac{3x^4+2}{x^2-5x+6} dx = \int_4^5 \left(3x^2 + 15x + 57\right) dx - 50 \int_4^5 \frac{dx}{x-2} + 245 \int_4^5 \frac{dx}{x-3}$$
$$= \frac{371}{2} - 50 \ln \left| \frac{5-2}{4-2} \right| + 245 \ln \left| \frac{5-3}{4-3} \right|$$
$$= \frac{371}{2} - 50 \ln \frac{3}{2} + 245 \ln 2$$

- $\int \frac{3x^2+2}{x^3-7x+6} dx$

  $$x^3 - 7x + 6 = (x-1)(x^2 + x - 6) = (x-1)(x-2)(x+3)$$

  Write:

  $$\frac{3x^2+2}{(x-1)(x-2)(x+3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x+3}$$
  $$3x^2 + 2 = A(x-2)(x+3) + B(x-1)(x+3) + C(x-1)(x-2)$$
  $$3 \cdot 1 + 2 = A(1-2)(1+3) + B \cdot 0 + C \cdot 0,\ A = -\frac{5}{4}$$
  $$3 \cdot 4 + 2 = A \cdot 0 + B(2-1)(2+3) + C \cdot 0, B = \frac{14}{5}$$
  $$3 \cdot 9 + 2 = A \cdot 0 + B \cdot 0\ + C(-3-1)(-3-2),\ C = \frac{29}{20}$$

$$\int \frac{3x^2+2}{x^3-7x+6}dx = -\frac{5}{4}\int \frac{dx}{x-1} + \frac{14}{5}\int \frac{dx}{x-2} + \frac{29}{20}\int \frac{dx}{x+3}$$
$$= -\frac{5}{4}\ln|x-1| + \frac{14}{5}\ln|x-2| + \frac{29}{20}\ln|x+3| + C$$

- $\int \frac{(x^2+1)dx}{x^3+7x^2+16x+12}$

  $$x^3+7x^2+16x+12 = (x+2)(x^2+5x+6) = (x+2)^2(x+3)$$

  Write:

  $$\frac{x^2+1}{(x+2)^2(x+3)} = \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{C}{x+3}$$

  Then

  $$x^2+1 = A(x+2)(x+3) + B(x+3) + C(x+2)^2$$
  $$x^2+1 = Ax^2+5Ax+6A+Bx+3B+Cx^2+4Cx+4C$$
  $$= (A+C)x^2 + (5A+B+4C)x + 6A+3B+4C$$

  So

  $$A+C = 1$$
  $$5A+B+4C = 0$$
  $$6A+3B+4C = 1$$

  Subtract 5 times equation 1 from eqaution 2 and 6 times equation 1 from equation 3 to get

  $$A+C = 1$$
  $$B-C = -5$$
  $$3B-2C = -5$$

  Subtract 3 times equation 2 form equation 3 to get

  $$A+C = 1$$
  $$B-C = -5$$
  $$C = 10$$

We then get iteratively $C = 10, B = 5$, and $A = -9$.
Alternatively

$$x^2 + 1 \;= A(x+2)(x+3) + B(x+3) + C(x+2)^2$$

Plug in $x = -2$

$$5 = 0 \cdot A + B(-2+3) + 0 \cdot C, B = 5$$

Plug in $x = -3$

$$10 = 0 \cdot A + 0 \cdot B + C(-3+2)^2, C = 10$$

Now

$$\begin{aligned} x^2 + 1 &= A(x+2)(x+3) + 5(x+3) + 10(x+2)^2 \\ x^2 + 1 - 5(x+3) - 10(x+2)^2 &= A(x+2)(x+3) \\ -9x^2 - 54 - 45x &= A(x+2)(x+3) \\ -9\,(x+3)\,(x+2) &= A(x+2)(x+3),\;\; A = -9 \end{aligned}$$

- $\int \frac{x+2}{x^4-1} dx$

$$x^4 - 1 = (x-1)(x+1)(x^2+1)$$

Write

$$\begin{aligned} \frac{x+2}{x^4-1} &= \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1} \\ x+2 &= A(x+1)(x^2+1) + B(x-1)(x^2+1) + (Cx+D)(x-1)(x+1) \\ x+2 &= (A+B+C)\,x^3 + (A-B+D)\,x^2 + (A+B-C)\,x + A - B - D \end{aligned}$$

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System of equations method:

$$\begin{aligned} x+2 &= (A+B+C)\,x^3 + (A-B+D)\,x^2 + (A+B-C)\,x + A - B - D \\ A+B+C &= 0 \\ A-B+D &= 0 \\ A+B-C &= 1 \\ A-B-D &= 2 \end{aligned}$$

Subtract equation 1 from equations 2,3,4 to get

$$\begin{aligned} A+B+C &= 0 \\ -2B-C+D &= 0 \\ -2C &= 1 \\ -2B-C-D &= 2 \end{aligned}$$

Subtract equation 2 from equation 4 to get

$$\begin{aligned} A+B+C &= 0 \\ -2B-C+D &= 0 \\ -2C &= 1 \\ -2D &= 2 \end{aligned}$$

Now we get $D=-1, C=-\frac{1}{2}, B=\frac{-1}{2}(1-\frac{1}{2})=-\frac{1}{4}$, $A=\frac{1}{4}+\frac{1}{2}=\frac{3}{4}$
Root sustitution method:

$$x+2 = A(x+1)(x^2+1)+B(x-1)(x^2+1)+(Cx+D)(x-1)(x+1)$$

$$-1+2 = A\cdot 0 + B(-1-1)(1+1) + (-C+D)\cdot$$

$$1+2 = A(1+1)(1+1) + B\cdot 0 + (C+D)\cdot 0, A$$

$$\begin{aligned} x+2-\frac{3}{4}(x+1)(x^2+1)+\frac{1}{4}(x-1)(x^2+1) &= (Cx+D)(x-1)(x+1) \\ -\frac{1}{2}x^3-x^2+\frac{1}{2}x+1 &= (Cx+D)(x-1)(x+1) \\ -\frac{1}{2}(x-1)(x+2)(x+1) &= (Cx+D)(x-1)(x+1), C=-\frac{1}{2}, D= \end{aligned}$$

$$\begin{aligned} \int \frac{x+2}{x^4-1}dx &= \frac{3}{4}\int\frac{dx}{x-1}-\frac{1}{4}\int\frac{dx}{x+1}-\frac{1}{2}\int\frac{(x+2)\,dx}{x^2+1} \\ &= \frac{3}{4}\ln|x-1|-\frac{1}{4}\ln|x+1|-\frac{1}{4}\int\frac{2xdx}{x^2+1}-\int\frac{dx}{x^2+1} \\ &= \frac{3}{4}\ln|x-1|-\frac{1}{4}\ln|x+1|-\frac{1}{4}\ln\left(x^2+1\right)-\arctan x+C \end{aligned}$$

- $\int_1^2 \frac{(x^2+1)}{(x+1)(x^2+4x+13)} dx$

$$\begin{aligned} \frac{x^2+1}{(x+1)(x^2+4x+13)} &= \frac{A}{x+1} + \frac{Bx+C}{x^2+4x+13} \\ x^2+1 &= A(x^2+4x+13) + (Bx+C)(x+1) \\ 1+1 &= A(1-4+13) + (-B+C)\cdot 0 \\ A &= \frac{1}{5} \end{aligned}$$

$$\begin{aligned} x^2+1-\frac{1}{5}(x^2+4x+13) &= (Bx+C)(x+1) \\ \frac{4}{5}\left(x+1\right)\left(x-2\right) &= (Bx+C)(x+1),\ B=\frac{4}{5},\ C=-\frac{8}{5} \end{aligned}$$

$$\begin{aligned} x^2+1 &= A(x^2+4x+13) + (Bx+C)(x+1) \\ x^2+1 &= \left(A+B\right)x^2 + \left(4A+B+C\right)x + 13A + C \\ A+B &= 1 \\ 4A+B+C &= 0 \\ 13A+C &= 1 \end{aligned}$$

$$\begin{aligned} A+B &= 1 \\ -3B+C &= -4 \\ -13B+C &= -12 \end{aligned}$$

Divide eqaution 2 by -3.

$$\begin{aligned} A+B &= 1 \\ B-\frac{C}{3} &= \frac{4}{3} \\ -13B+C &= -12 \end{aligned}$$

Add 13 times equation 2 to eqaution 3

$$A + B = 1$$
$$B - \frac{C}{3} = \frac{4}{3}$$
$$(1 - \frac{13}{3})C = -12 + \frac{52}{3}, -\frac{10}{3}C = \frac{16}{3}$$
$$C = -\frac{8}{5}, B = \frac{4}{3} - \frac{1}{3}\frac{8}{5} = \frac{4}{5}, A = 1 - \frac{4}{5} = \frac{1}{5}$$

$$\begin{aligned}
\int_1^2 \frac{(x^2+1)dx}{(x+1)(x^2+4x+13)}dx &= \frac{1}{5}\int_1^2 \frac{dx}{x+1} + \frac{4}{5}\int_1^2 \frac{x-2}{x^2+4x+13}dx \\
&= \frac{1}{5}\int_1^2 \frac{dx}{x+1} + \frac{2}{5}\int_1^2 \frac{2x+4}{x^2+4x+13}dx + \frac{4}{5}\int_1^2 \frac{4}{x^2+4x+13}dx \\
&= \frac{1}{5}\int_1^2 \frac{dx}{x+1} + \frac{2}{5}\int_1^2 \frac{d(x^2+4x+13)}{x^2+4x+13} + \frac{4}{5}\int_1^2 \frac{4}{(x+2)^2+9}dx \\
&= \frac{1}{5}\left[\ln|x+1|\right]_1^2 + \frac{2}{5}\left[\ln\left|x^2+4x+13\right|\right]_1^2 + \frac{2}{5}\left[\frac{1}{3}\arctan\frac{x+}{3} \\
&= \frac{1}{5}\ln\left(\frac{3}{2}\right) + \frac{2}{5}\ln\left(\frac{25}{18}\right) + \frac{2}{15}\arctan\left(\frac{4}{3}\right) - \frac{2}{15}\arctan\left(\frac{3}{3}\right. \\
&\frac{1}{5}\ln\left(\frac{3}{2}\right) + \frac{2}{5}\ln\left(\frac{25}{18}\right) + \frac{2}{15}\arctan\left(\frac{4}{3}\right) - \frac{\pi}{30}
\end{aligned}$$

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