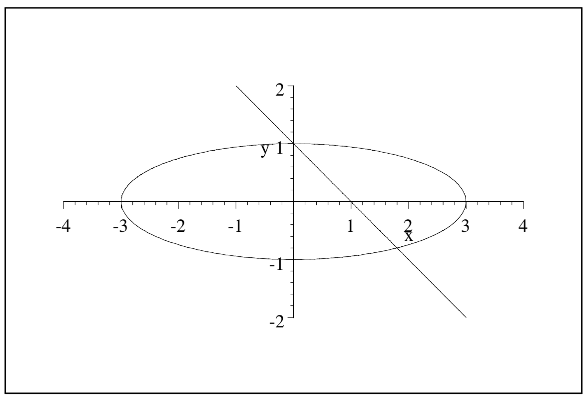
Calculus II - Test #3 Review Answers

Professor Broughton January, 2003

1. Integrals

1.a Sketch the area contained between the two curves

$$\begin{aligned} x^2 + 9y^2 &= 9 \\ x + y &= 1 \end{aligned}$$



1.b Find all points of intersection between the two curves:

$$\begin{aligned} y &= 1 - x \\ x^2 + 9(1-x)^2 &= 9 \\ 10x^2 - 18x &= 0 \\ 2x\,(5x - 9) &= 0 \\ x &= 0, \frac{9}{5} \\ (x, y) &= (0, 1), (\frac{9}{5}, \frac{-4}{5}) \end{aligned}$$

1.c Set up the integral(s) to compute the area between the two curves. You may integrate along either x or y axis:

$$\int_{-4/5}^{1} \left(3\sqrt{1-y^2} - (1-y)\right) dy$$

$$\int_{0}^{9/5} \left(\frac{1}{3}\sqrt{9-x^2} - (1-x)\right) dx + \int_{9/5}^{3} \left(\frac{2}{3}\sqrt{9-x^2}\right) dx$$

1.d What is the area, to 4 decimal places. Say how you got the area.:

$$\int_{-4/5}^{1} \left(3\sqrt{1-y^2} - (1-y)\right) dy$$

$$= -\frac{9}{10} + \frac{3}{4}\pi + \frac{3}{2}\arcsin\frac{4}{5} = 2.8471$$

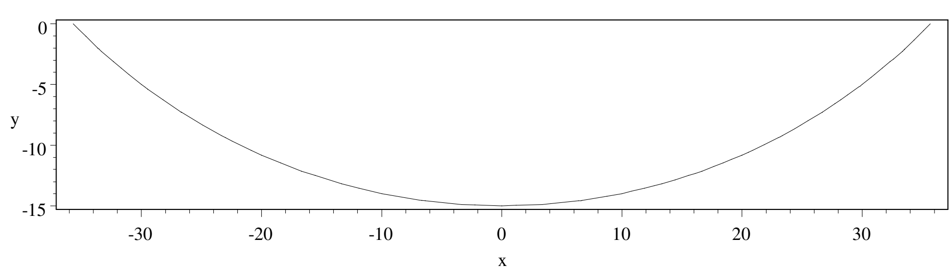
$$\int_{0}^{9/5} \left(\frac{1}{3}\sqrt{9-x^2} - (1-x)\right) dx + \int_{9/5}^{3} \left(\frac{2}{3}\sqrt{9-x^2}\right) dx$$

$$= -\frac{9}{10} - \frac{3}{2}\arcsin\frac{3}{5} + \frac{3}{2}\pi = 2.8471$$

2 A waste lagoon is modeled as a lower portion of a sphere, and has a vertical cross-section as given in the figure below. The cross-section is given by the portion of the circle,

$$x^2 + (y-35)^2 = 2500$$

lying below the x-axis. The dimensions are in feet.



a What is the volume of water in the lagoon if height of the water is 5 ft from the top. Express the volume as an integral and then evaluate the integral. What about 10 ft from the top?

	Integral	Volume
5 ft	$\pi \int_{-15}^{-5} (2500 - (y-35)^2)\, dy$	$\frac{14000}{3}\pi$
10 ft	$\pi \int_{-15}^{-10} (2500 - (y-35)^2)\, dy$	$\frac{3625}{3}\pi$

1. b The lagoon is being pumped out at 25 cu ft per minute until the water level reduced to 10 feet below the top. How many hours does this take?

3 Compute these limits. Show all steps.

$$\begin{aligned} \Delta V &= \frac{14000}{3}\pi - \frac{3625}{3}\pi = \frac{10375}{3}\pi \\ \text{time in hours} &= \frac{\Delta V}{25} \times \frac{1}{60} = \frac{10375}{3}\pi \times \frac{1}{25 \times 60} \\ &= \frac{83}{36}\pi = 7.24 \text{ hours} \end{aligned}$$

a

$$\begin{aligned} \lim_{x\to 5} \frac{x^2-25}{\cos(x-5)-1} &= \lim_{x\to 5} \frac{2x}{-\sin(x-5)} = \frac{10}{0} = \text{undefined} \\ \lim_{x\to 5} \frac{(x-5)^2}{\cos(x-5)-1} &= \lim_{x\to 5} \frac{2(x-5)}{-\sin(x-5)} = \lim_{x\to 5} \frac{2}{-\cos(x-5)} = -2 \end{aligned}$$

b

$$\lim_{x\to 1} \frac{\ln x}{x-1} = \lim_{x\to 1} \frac{1/x}{x} = \lim_{x\to 1} \frac{1}{x^2} = 1$$

c

$$\lim_{x\to\infty} \frac{x^3+3x}{e^{2x}} = \lim_{x\to\infty} \frac{3x^2+3}{2e^{2x}} = \lim_{x\to\infty} \frac{6x}{4e^{2x}} = \lim_{x\to\infty} \frac{6}{8e^{2x}} = \frac{6}{\infty} = 0$$

d

$$\lim_{x\to\infty} \frac{\ln x}{x^2} = \lim_{x\to\infty} \frac{1/x}{2x} = \lim_{x\to\infty} \frac{1}{2x^2} = \frac{1}{\infty} = 0$$

e

$$\lim_{x\to\infty} x^2 e^{-sx}, s > 0$$

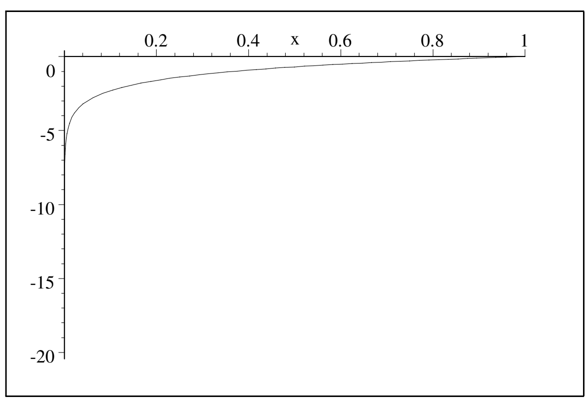
$$\lim_{x\to\infty} x^2 e^{-sx} = \lim_{x\to\infty} \frac{x^2}{e^{sx}} = \lim_{x\to\infty} \frac{2x}{se^{sx}} = \lim_{x\to\infty} \frac{2}{s^2e^{sx}} = \lim_{x\to\infty} \frac{2}{\infty} = 0$$

4. Compute these integrals showing them as a limit first, then compute the limit. Make a sketch of the graph showing the infinite limit or the vertical asymptotes.

a

$$\begin{aligned}\int_0^\infty t^2 e^{-st} dt &= \lim_{T\to\infty} \int_0^T t^2 e^{-st} dt \\ &= \lim_{T\to\infty} \left[-\frac{1}{s^3} \left(s^2 t^2 e^{-st} + 2ste^{-st} + 2e^{-st} \right) \right]_0^T \\ &= \lim_{T\to\infty} \frac{2 - e^{-sT} s^2 T^2 - 2e^{-sT} sT - 2e^{-sT}}{s^3} \\ &= \frac{2}{s^3}\end{aligned}$$

b



$$\begin{aligned}\int_0^1 \ln x dx &= \lim_{\epsilon\to 0^+} \int_\epsilon^1 \ln x dx = \lim_{\epsilon\to 0^+} [x\ln x - x]_\epsilon^1 \\ &= \lim_{\epsilon\to 0^+} (-1 - \epsilon\ln\epsilon + \epsilon) \\ &= -1 - \lim_{\epsilon\to 0^+} \frac{\ln\epsilon}{1/\epsilon} = -1 - \lim_{\epsilon\to 0^+} \frac{1/\epsilon}{-1/\epsilon^2} \\ &= -1 + \lim_{\epsilon\to 0^+} \epsilon = -1\end{aligned}$$

c

$$\begin{aligned}
\int_1^\infty x^p dx, p &> 1 \\
\int_1^\infty x^p dx &= \lim_{T\to\infty} \int_1^T x^p dx \\
&= \lim_{T\to\infty} \left[\frac{x^{p+1}}{p+1}\right]_0^T = \lim_{T\to\infty} \left(\frac{T^{p+1}-1}{p+1}\right) = \infty \\
\int_1^\infty x^{-p} dx, p &> 1 \\
\int_1^\infty x^{-p} dx &= \lim_{T\to\infty} \int_1^T x^p dx \\
&= \lim_{T\to\infty} \left[\frac{x^{1-p}}{1-p}\right]_0^T = \lim_{T\to\infty} \left(\frac{T^{-(p-1)}-1}{1-p}\right) = \frac{1}{p-1}
\end{aligned}$$

d

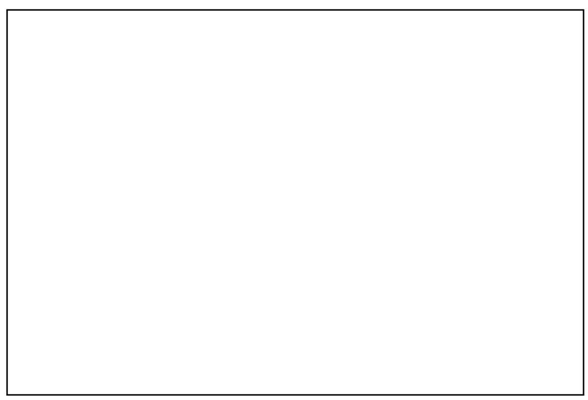
$$\begin{aligned}
\int_{-\infty}^\infty \frac{dx}{1+x^2} &= \int_{-\infty}^0 \frac{dx}{1+x^2} + \int_0^\infty \frac{dx}{1+x^2} \\
&= 2\int_0^\infty \frac{dx}{1+x^2} = 2\lim_{T\to\infty} \int_0^T \frac{dx}{1+x^2} \\
&= 2\lim_{T\to\infty} [\arctan T]_0^T = 2\lim_{T\to\infty} \arctan T \\
&= 2\frac{\pi}{2} = \pi
\end{aligned}$$

$$\arctan T$$

$\arctan T$ and $\frac{\pi}{2}$

e

$$\frac{1}{(1-x)^2}$$



$$\begin{aligned}
\int_0^2 \frac{dx}{(1-x)^2} &= \int_0^1 \frac{dx}{(1-x)^2} + \int_1^2 \frac{dx}{(1-x)^2} \\
&= \lim_{\epsilon \to 0^+} \int_0^{1-\epsilon} \frac{dx}{(1-x)^2} + \lim_{\epsilon \to 0^+} \int_{1+\epsilon}^2 \frac{dx}{(1-x)^2} \\
&= \lim_{\epsilon \to 0^+} \left[\frac{-1}{1-x} \right]_0^{1-\epsilon} + \lim_{\epsilon \to 0^+} \left[\frac{-1}{1-x} \right]_{1+\epsilon}^2 \\
&= \lim_{\epsilon \to 0^+} \frac{1-\epsilon}{\epsilon} + \lim_{\epsilon \to 0^+} \frac{1-\epsilon}{\epsilon} = \frac{1}{0^+} + \frac{1}{0^+} = \infty
\end{aligned}$$

f Another similar example

$$\begin{aligned}
\int_1^2 \frac{dx}{\sqrt{x-1}} &= \lim_{\epsilon \to 0^+} \int_{1+\epsilon}^2 \frac{dx}{\sqrt{x-1}} \\
&= \lim_{\epsilon \to 0^+} \left[2\sqrt{x-1}\right]_{1+\epsilon}^2 \\
&= \lim_{\epsilon \to 0^+} \left[2\sqrt{x-1}\right]_{1+\epsilon}^2 \\
&= \lim_{\epsilon \to 0^+} \left(2 - 2\sqrt{\epsilon}\right) \\
&= 2
\end{aligned}$$