

CHAPTER 6

SYSTEM FREQUENCY RESPONSE

Finding the frequency response of systems is important in many fields of engineering and science—wherever dynamic systems are encountered. From vibration analysis in mechanical engineering to filter design in electrical engineering to investigations of the response of the human ear in biomedical engineering, the frequency response of a dynamical system is found to be an important analysis and design tool. The utility of frequency-domain analysis, when combined with the Fourier techniques discussed in this chapter, can hardly be overstated.

The present methods used in frequency domain analysis have grown out of the work of Arthur Edwin Kennelly who presented a paper on “Impedance” to the American Institute of Electrical Engineers in 1893. His paper represented the first published use of complex numbers in alternating current circuit theory. Later in the same year, Charles Proteus Steinmetz discussed the significance of Kennelly’s paper before the International Electrical Congress in Chicago. Steinmetz introduced the engineering community the utility of the $j\omega$ notation in phasor analysis. His first textbook, *Theory and Calculation of AC Phenomena*, was the first of many textbooks that he wrote.

Steinmetz worked at General Electric for several years. A well known story about Steinmetz involved a \$10,000 consulting bill he sent to General Electric in 1902 after he had retired from the firm. The consultation involved trouble shooting a complex system which had failed, and his work involved a chalk mark showing where the fault lay. Taken aback by the amount of the bill, General Electric asked for an itemized list of charges. The invoice Steinmetz sent appears below.

1. Making the chalk mark \$1.
2. Knowing where to make it \$9,999.

A.E. Kennelly was professor of electrical engineering at Harvard University from 1902-1930 and professor of electrical engineering at MIT from 1913 to 1924. C.P. Steinmetz’s work in AC machines and power resulted in many significant papers and patents. In 1895, he patented a “system of distribution by alternating current.”

The revolutionary techniques of Fourier Analysis were born in 1807 when Joseph Fourier submitted a paper on the analytic theory of heat to the French Academy of Science. In this paper Fourier Analysis was first outlined, but, instead of greeting the work as one of genius, the commission examining the work—Laplace, Lagrange, Biot, and Poisson attacked the work as lacking rigor. Even the passage of four years were insufficient to satisfy Laplace and Lagrange as to the validity of Fourier’s methods when, in 1811, Fourier won the Academy’s grand prize in mathematics.

6.1 System Frequency Response

When referring to a system’s frequency response, the system is considered to provide a relationship between an input and output. The system input is the source of excitation and the system output is whichever system variable is of interest.



Figure 6.1: Dynamical System

In sinusoidal steady-state analysis, the input is assumed to be a sinusoid. The system frequency response is defined by the **sinusoidal steady-state response** to the **sinusoidal input**. In finding a system’s frequency response, the frequency is left as a variable. Since the relation which results holds for all frequencies, the system frequency response contains an enormous amount of information. The system frequency response contains not just information of the system’s response at a particular frequency, but can contain information at all frequencies of interest.

Since a system’s frequency response involves find the sinusoidal steady-state response of a system, the techniques can be considered to be applications of phasor analysis.

Example 6.1

Consider the system shown below, a force exerted on a mass that moves on a platform with viscous damping. Let the output—that is, the system variable of interest—be the resulting speed.

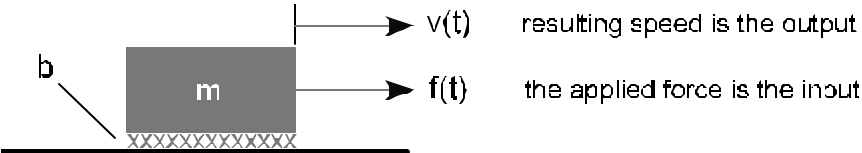


Figure 6.2: Moving Mass with Viscous Damping

From Newton’s 2nd Law

$$m \frac{dv(t)}{dt} = f(t) - bv(t)$$

$$m \frac{dv(t)}{dt} + bv(t) = f(t)$$

This is the time-domain system equation-of-motion (EOM). To find the system frequency response, we first assume $f(t)$ is a sinusoid, $f(t) = F \cos(\omega t + q_f)$.

The corresponding phasor equation is $m(j\omega)\mathbf{V} + b\mathbf{V} = \mathbf{F}$

Note: In assuming RMS values for phasor quantities, electrical systems are a notable exception. In other areas of engineering and science, peak values of phasors are commonly used. In this equation, let us take \mathbf{V} and \mathbf{F} to be peak phasors (the frequency-domain variables) corresponding to time-domain variables $v(t)$ and $f(t)$.

$$\begin{aligned} v(t) &= V \cos(\omega t + q_v) & \Leftrightarrow & \mathbf{V} = V \angle q_v \\ f(t) &= F \cos(\omega t + q_f) & \Leftrightarrow & \mathbf{F} = F \angle q_f \end{aligned}$$

Note that the differential EOM in the time domain has been transformed into an algebraic equation in the frequency domain.

Let $m = 0.25 \text{ kg}$ and $b = 3 \text{ Ns/m}$. Solving for \mathbf{V} gives

$$\mathbf{H} = \frac{\mathbf{F}}{j\omega m + b} = \frac{\mathbf{F}}{j\omega(0.25 \text{ kg}) + 3 \text{ Ns/m}}$$

The system frequency response, \mathbf{H} , is expressed as a ratio of the output to the input.

$$\begin{aligned} \mathbf{H} &= \frac{\mathbf{V}}{\mathbf{F}} = \frac{1}{j\omega(0.25 \text{ kg}) + 3 \text{ Ns/m}} \\ \mathbf{H} &= H \angle q_H = \frac{1}{\sqrt{[(0.25 \omega)^2 + 3^2]}} \angle \tan^{-1}\left(\frac{0.25\omega}{3}\right) \end{aligned}$$

If the system frequency response is known, the input and output can be determined once the other is known. For example, in the example above, take

$$f(t) = 20 \cos(16t + 30^\circ) \text{ N}$$

Noting that $\omega = 16 \text{ r/s}$,

$$\mathbf{H} = H \angle q_H = \frac{1}{\sqrt{[4^2 + 3^2]}} \angle \tan^{-1}\left(\frac{4}{3}\right) \cong 0.2 \angle 53.1^\circ \frac{\text{m/s}}{\text{N}}$$

From the input peak phasor, $\mathbf{F} = 20 \angle 30^\circ \text{ N}$, and \mathbf{H} , the output peak phasor, \mathbf{V} , can readily be determined.

$$\mathbf{V} = \mathbf{F} \mathbf{H} = (20 \angle 30^\circ \text{ N}) \left(0.2 \angle 53.1^\circ \frac{\text{m/s}}{\text{N}} \right) = 4 \angle 83.1^\circ \text{ m/s}$$

From \mathbf{V} , the time-domain speed is readily seen to be $v(t) = 4 \cos(16t + 83.1^\circ) \text{ m/s}$.

Note: This is **not** the complete response. Sinusoidal steady-state analysis only finds the steady-state solution. The natural or transient solution is not found.

Example 6.2

Find the frequency response of the electrical system below. The voltage source is the system input, and the output is $i(t)$.

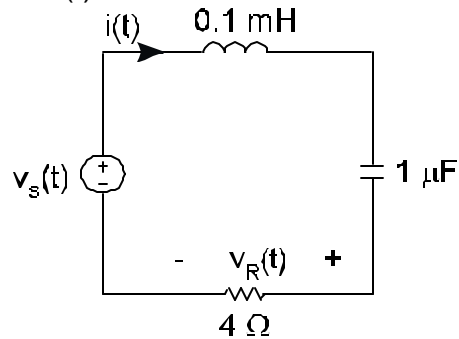


Figure 6.3: LRC Series Circuit

From KVL, we can obtain the time-domain EOM.

$$v_s(t) = L \frac{di(t)}{dt} + \frac{1}{C} \int_{t'=-\infty}^t i(t') dt' + R i(t) = 10^{-4} \text{H} \frac{di(t)}{dt} + \frac{1}{10^{-6} \text{F}} \int_{t'=-\infty}^t i(t') dt' + 4 \Omega i(t)$$

Taking derivatives to eliminate the integral and dropping element units for clarity,

$$\frac{dv_s(t)}{dt} = 10^{-4} \frac{d^2i(t)}{dt^2} + \frac{1}{10^{-6}} i(t) + 4 \frac{di(t)}{dt}$$

$$j\omega V_s = (j\omega)^2 10^{-4} I + 10^6 I + j\omega 4 I$$

where the phasors are understood to be in RMS for this electrical system. The system frequency response is found by taking the ratio of the output (the current) to the input (the source voltage).

$$H = \frac{I}{V_s} = \frac{j\omega}{(j\omega)^2 10^{-4} + 10^6 + j\omega 4}$$

Let us take another look at this example. The steps of finding the time-domain EOM, taking a derivative, and then finding the corresponding phasor equation can be simplified by using the phasor circuit.

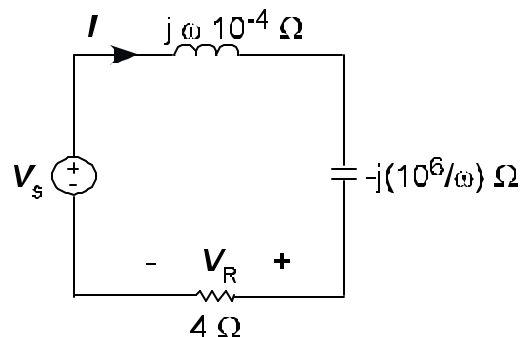


Figure 6.4: Phasor Circuit

Where V_s and I are understood to be RMS phasors. KVL gives the frequency-domain EOM.

$$V_s = j\omega 10^{-4} I - j \frac{10^6}{\omega} I + 4I$$

Multiplying by $j\omega$ results in the same equation as found above.

$$j\omega V_s = (j\omega)^2 10^{-4} I + 10^6 I + j\omega 4I$$

$$H = \frac{I}{V_s} = \frac{j\omega}{(j\omega)^2 10^{-4} + 10^6 + j\omega 4}$$

Example 6.3

The most convenient technique with which to find the system frequency response, and the one used in the rest of this chapter, is to use Laplace transforms. Since the system frequency response finds only the sinusoidal steady-state response, we can ignore initial conditions since they do not affect the steady-state response.

Let us use the same system as used in the previous example.

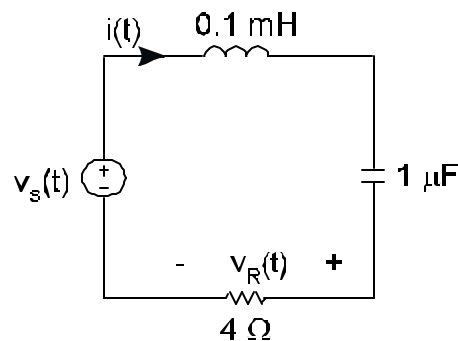


Figure 6.5: LRC Series Circuit

The time-domain EOM is

$$v_s(t) = 10^{-4} \frac{di(t)}{dt} + \frac{1}{10^{-6}} \int_{t'=-\infty}^t i(t') dt' + 4i(t)$$

Taking the Laplace transform (ICs assumed to be zero) of this equation gives.

$$V_s(s) = \mathcal{L} [v_s(t)]_{i_C(s)=0} = 10^{-4} sI(s) + 10^6 \frac{I(s)}{s} + 4I(s)$$

Multiplying by s and taking the ratio between $I(s)$ and $V_s(s)$, we obtain the system transfer function $H(s)$.

$$sV_s(s) = 10^{-4}s^2I(s) + 10^6I(s) + 4sI(s)$$

$$H(s) = \frac{I(s)}{V(s)} = \frac{s}{10^{-4}s^2 + 10^6 + 4s}$$

The system frequency response, H , results when s is replaced by $j\omega$ in the system transfer function, $H(s)$.

$$H = \frac{I}{V_s} = H(s)|_{s \rightarrow j\omega}$$

Note: The question naturally arises about whether I and V_s in the relation above need to be in RMS or peak. The answer is that it does not matter as long as they are both peak or both RMS. Since H is a ratio between two phasors, it will be unaltered as long as both phasors are either peak or both RMS. We will adopt the convention of using RMS for electrical systems and peak for others.

6.2 s-Domain Analysis

In electrical systems, one usually does not find the time-domain equations and then take the Laplace transform. More typically, the s-domain circuit is found, and s-domain analysis is performed using this circuit. The strategy is much like that in phasor analysis. In s-domain analysis, the Laplace transforms of the element relations are found. This results in algebraic relationships between the s-domain voltages and currents. The ratio of the s-domain voltage across an element to the s-domain current through it—passive sign convention (PSC) assumed—is the s-domain impedance.

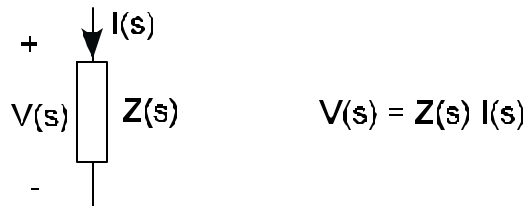


Figure 6.6: Impedance

The s-domain element relations for the resistance, inductance, and capacitance are shown below.

$$\mathbf{L}_{i_c=0} [v(t) = L \frac{di(t)}{dt}] \rightarrow V(s) = sL I(s)$$

$$\mathbf{L}_{i_c=0} [i(t) = C \frac{dv(t)}{dt}] \rightarrow I(s) = sC V(s)$$

$$\mathbf{L}_{i_c=0} [v(t) = R i(t)] \rightarrow V(s) = R I(s)$$

The resulting s-domain impedances are

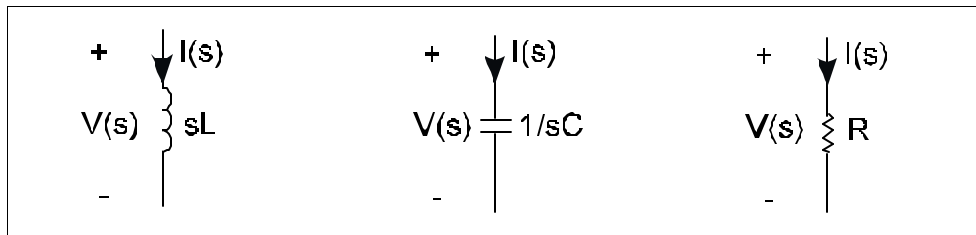


Figure 6.7: Impedance of Inductance, Capacitance, and Resistance

Example 6.4

Using the same circuit as in the previous two examples, the time-domain and s-domain circuits are shown below.

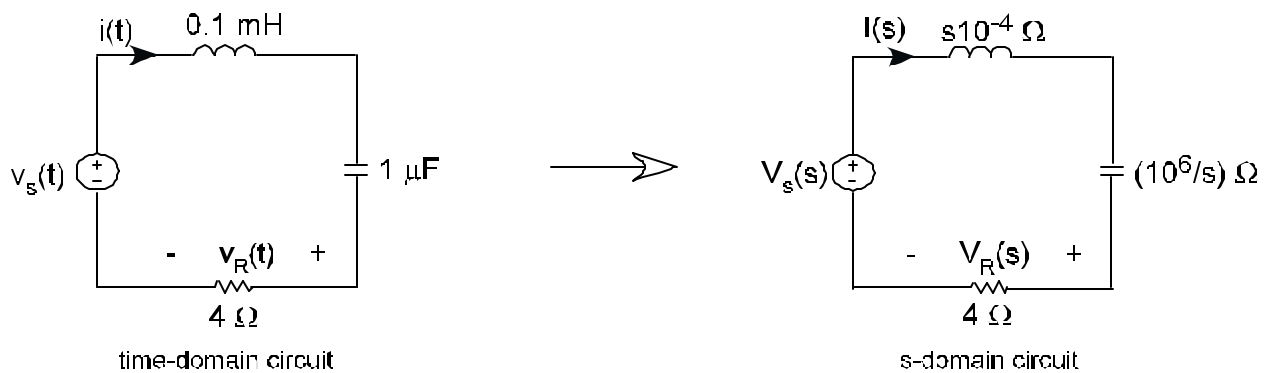


Figure 6.8: Using Laplace Transforms in Circuit Analysis

KVL gives the s-domain equation of motion. Taking the ratio of $I(s)$ to $V_s(s)$ gives the system transfer function.

$$V_s(s) = s10^{-4}I(s) + \frac{10^6}{s}I(s) + 4I(s)$$

$$H(s) = \frac{I(s)}{V_s(s)} = \frac{1}{s10^{-4} + \frac{10^6}{s} + 4} = \frac{s}{s^210^{-4} + 10^6 + 4s}$$

Substituting s with $j\omega$ gives the same system frequency response as before.

$$H = \frac{I}{V_s} = H(s)|_{s \rightarrow j\omega} = \frac{j\omega}{(j\omega)^2 10^{-4} + 10^6 + 4j\omega}$$

Note: As noted above I and V_s can be either peak or RMS as long as both are the same, that is, as long as both the input and output are either are RMS or else both peak. One cannot be peak and the other RMS.

Suppose that the output is taken to be $v_R(t)$, the voltage across the 4Ω resistance. The system frequency response for this system would be as shown below.

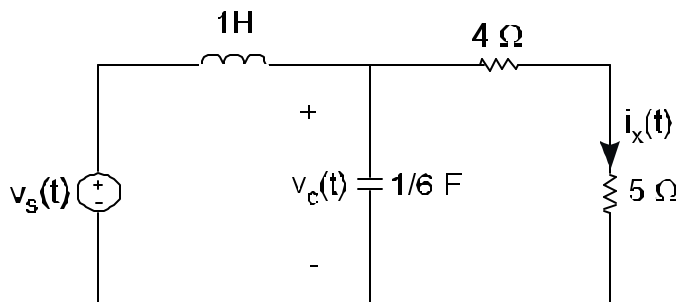
$$H_R = \frac{V_R}{V_s} = \frac{4\Omega I}{V_s} = \frac{4\Omega I(s)}{V_s(s)} \Big|_{s \rightarrow j\omega} = \frac{4s}{s^2 10^{-4} + 10^6 + 4s} \Big|_{s \rightarrow j\omega}$$

$$H_R = \frac{4j\omega}{(j\omega)^2 10^{-4} + 10^6 + 4j\omega}$$

When combined with nodal analysis this technique becomes a very powerful and general method of finding the system frequency response in any linear lumped-element electrical circuit.

Example 6.5

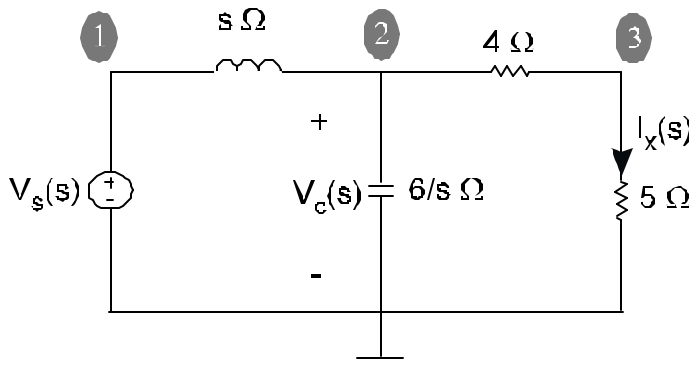
Find the frequency response of this system.



time-domain circuit

Figure 6.9: Time-Domain Circuit

The first step is to produce the s-domain circuit.



s-domain circuit

Figure 6.10: s-Domain Circuit

The nodal equations read

- 1) $V_1 = V_s$
- 2) $\frac{V_2 - V_1}{s} + \frac{V_2}{6/s} + \frac{V_2 - V_3}{4} = 0$
- 3) $\frac{V_3 - V_2}{4} + \frac{V_3}{5} = 0$

Solving these equations for $V_1(s)$, $V_2(s)$, and $V_3(s)$ in terms of the source voltage, $V_s(s)$:

$$V_1(s) = V_s(s)$$

$$V_2(s) = \frac{18 V_s(s)}{3s^2 + 2s + 18}$$

$$V_3(s) = \frac{10 V_s(s)}{3s^2 + 2s + 18}$$

From nodal analysis, we know that any circuit variable (any voltage or current) can be expressed in terms of node voltages. To complete the process, express the output in terms of node voltages, and find the system transfer function by taking the ratio of this output to the input source.

For the first example, consider $V_c(s)$ as the output. Expressing $V_c(s)$ in terms of node voltages, one obtains, $V_c(s) = V_2(s)$

$$H_c(s) = \frac{V_c(s)}{V_s(s)} = \frac{18}{3s^2 + 2s + 18}$$

$$H_c = H_c(s)|_{s \rightarrow j\omega} = \frac{18}{3(j\omega)^2 + 2j\omega + 18}$$

As another example, consider $I_x(s)$ as the output

$$H_x(s) = \frac{I_x(s)}{V_s(s)} = \frac{V_3(s)/5}{V_s(s)} \frac{2}{3s^2 + 2s + 18}$$

$$H_x = H_x(s)|_{s \rightarrow j\omega} = \frac{2}{3(j\omega)^2 + 2j\omega + 18}$$

As one further example, suppose $v_s(t) = 10 \cos(2t - 10^\circ)$ V, find the resulting $v_c(t)$ in steady state.

$$H_{c(\omega=2r/s)} = \left. \frac{V_c}{V_s} \right|_{\omega=2r/s} = \frac{18}{3(j2)^2 + 2j2 + 18} = 2.50 \angle -33.7^\circ$$

Where V_c and V_s are assumed to be RMS since this is an electrical systems problem.

$$V_c = H_c V_s \cong 2.50 \angle -33.7^\circ \left(\frac{10}{\sqrt{2}} \angle 10^\circ \text{ V} \right) = 17.7 \angle -23.7^\circ \text{ V}$$

$$v_c(t) = 25 \cos(2t - 23.7^\circ) \text{ V}$$

6.3 Bode Plots

The system transfer function that relates an input to an output is a complex number. In polar form, $\mathbf{H} = H\angle\theta_H$. Both the magnitude, H , and the phase, θ_H , are functions of frequency. Showing the system frequency response requires two graphs: one to show how the magnitude varies with frequency and a second to show how the phase varies with frequency.

Bode plots are a widely used method of graphically displaying the frequency response of systems—the Bode magnitude plot giving magnitude information and the Bode phase plot giving phase information.

The Bode magnitude plot shows magnitude versus frequency. The magnitude is in decibels, and the frequency is logarithmically scaled and can be either angular frequency (radian/sec) or linear frequency (Hertz). Decibels are defined as

$$H_{\text{dB}} = 20 \log_{10} |\mathbf{H}| = 20 \log_{10} H$$

Note that the magnitude, H , can be readily recovered from H_{dB} since $H = 10^{\frac{H_{\text{dB}}}{20}}$.

The Bode phase plot shows phase versus frequency. The phase is typically in degrees, but could be given in radians. Like the Bode magnitude plot, the frequency is logarithmically scaled in either radian/sec or in Hertz.

Recall that, in s-domain analysis, the input and output are related via the system transfer function. To be specific, consider the input and output to both be voltages (of course, depending on the system, they could be temperatures, currents, forces...).

$$V_o(s) = H(s) V_i(s)$$

The sinusoidal steady-state response is given when $s \rightarrow j\omega$. When s is replaced by $j\omega$, that V_o , TF, and V_i all become complex functions of ω .

$$\mathbf{V}_o(\omega) = \mathbf{H}(\omega) \mathbf{V}_i(\omega)$$

Think of these phasor quantities in polar form.

$$\begin{aligned} V_o\angle q_o &= H\angle q_{\text{TF}} V_i\angle q_i \\ H\angle q_{\text{TF}} &= \frac{V_o\angle q_o}{V_i\angle q_i} = \frac{V_o}{V_i} \angle (q_o - q_i) \end{aligned}$$

Example 6.6

Suppose a system has the frequency response shown below.

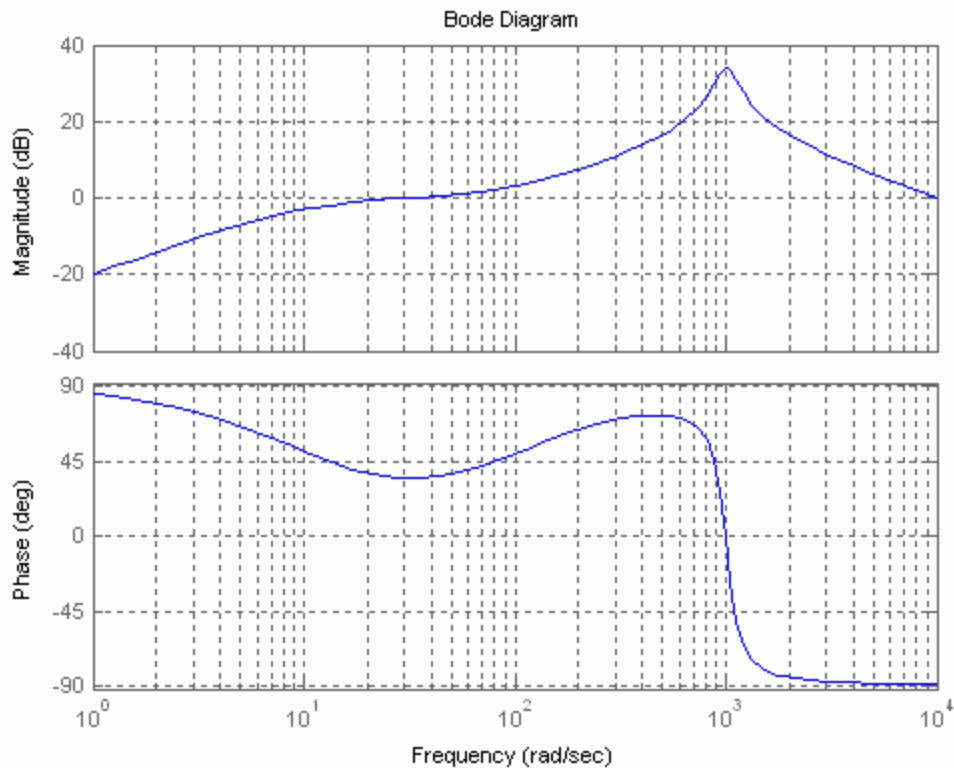


Figure 6.11: System Frequency Response using MATLAB

Assuming the frequency response is the ratio between two voltages, what is the output voltage in sinusoidal steady-state if the input is $V_i = 10 \cos(500t - 15^\circ)$ volts?

Evaluating the Bode plots at $\omega = 500$ r/s, we can estimate the system frequency response's magnitude and phase.

$$H_{dB}(\omega = 500 \text{ r/s}) \cong 17.5 \text{ dB} \Rightarrow H = |H| \cong 10^{\frac{17.5}{20}} = 7.5$$

$$\angle H(\omega = 500 \text{ r/s}) \cong 63^\circ$$

Now, knowing $\mathbf{H} = H \angle \theta_H$, we can find the output through the relation $\mathbf{V}_o = \mathbf{H} \mathbf{V}_i$. In polar form this relation is

$$\begin{aligned} V_o \angle \theta_o &= H \angle \theta_H V_i \angle \theta_i \\ V_o &= H V_i \quad \text{and} \quad \theta_o = \theta_H + \theta_i \end{aligned}$$

Using RMS phasors for the voltages,

$$\mathbf{V}_o = V_o \angle \theta_o \cong (7.5 \angle 63^\circ)(7.071 \angle -15^\circ \text{V}) = 53.03 \angle 48^\circ \text{V}$$

From V_o , $v(t)$ can readily be determined.

$$v_o(t) = 75 \cos(500t + 48^\circ) \text{ V}$$

Note: 1) Peak valued phasors could have just as easily been used rather than RMS. In fact, if we were interested in $v_o(t)$, using RMS phasors merely serves to force us to divide and then multiply by $\sqrt{2}$. If one were only interested in $v_o(t)$, the amplitude of 75 V could have been calculated more straightforwardly by just multiplying the input amplitude, 10 V, by the magnitude of H at $\omega = 500$ r/s.

The only reason for the convention of using RMS in frequency domain analysis is for the power relation to be $S = VI^*$ (for V and I RMS phasors) rather than $S = \frac{1}{2}VI^*$ (for V and I peak phasors).

2) In this case, the transfer function, H , has no units since it is the ratio of two voltages. This need not be the case. The units of H are the units of the output divided by the units of the input. For example, if the input were a current and the output a voltage, then the units of H would be that of voltage divided by current—Ohms.

The system transfer function can also be used in determining the input if the output is known. For example, suppose the output voltage of the system is measured in the laboratory to be $30 \cos(80t)$ mV. What input voltage would this imply? Evaluating the

Bode plots at $\omega = 80$ r/s, $H \cong 10^{\frac{7.5}{20}} \angle 27^\circ = 2.37 \angle 27^\circ$

$$V_i \cong \frac{V_o}{H} = \frac{30/\sqrt{2} \angle 0^\circ \text{ V}}{2.37 \angle 27^\circ} = 8.95 \angle -27^\circ \text{ V}$$

$$v(t) \cong 8.95\sqrt{2} \cos(80t - 27^\circ) \text{ V} = 12.7 \cos(80t - 27^\circ) \text{ V}$$

Sketching Bode Plots

Begin with the s-domain transfer function. The transfer function can be written as a ratio of polynomials in s.

$$H(s) = \frac{a_n s^z + b_n s^{z-1} + \dots}{a_d s^p + b_d s^{p-1} + \dots}$$

The coefficients are just parameters of the system and so are real numbers. For polynomials with real coefficients, their roots are either real or occur in complex conjugate pairs. For this case, the transfer function can be manipulated in the form below. This form is called Bode form.

$$H(s) = K_b \frac{s^n \left(\frac{s}{w_{bz1}} + 1 \right) \dots \left(\frac{s^2}{w_{nz1}^2} + \frac{2z_{z1}}{w_{nz1}} s + 1 \right) \dots}{s^m \left(\frac{s}{w_{bp1}} + 1 \right) \dots \left(\frac{s^2}{w_{np1}^2} + \frac{2z_{p1}}{w_{np1}} s + 1 \right) \dots}$$

To obtain the system frequency response, s is replaced by $j\omega$. When s is replaced by $j\omega$, the output-input ratio becomes complex.

$$H = K_b \frac{(j\omega)^n \left(\frac{j\omega}{\omega_{bz1}} + 1\right) \dots \left(\frac{(j\omega)^2}{\omega_{nz1}^2} + \frac{2z_{z1}}{\omega_{nz1}} j\omega + 1\right) \dots}{(j\omega)^m \left(\frac{j\omega}{\omega_{bp1}} + 1\right) \dots \left(\frac{s^2}{\omega_{np1}^2} + \frac{2z_{p1}}{\omega_{np1}} j\omega + 1\right) \dots}$$

Bode Magnitude Plots

$$H_{db} = 20 \log_{10} |H|$$

$$H_{db} = 20 \log_{10} |K_b| + 20(n-m) \log_{10} |j\omega| + 20 \log \left| \frac{j\omega}{\omega_{bz1}} + 1 \right| + \dots$$

$$+ 20 \log \left| \frac{(j\omega)^2}{\omega_{nz1}^2} + \frac{2z_{z1}}{\omega_{nz1}} j\omega + 1 \right| + \dots - 20 \log \left| \frac{j\omega}{\omega_{bp1}} + 1 \right| - \dots$$

$$- 20 \log \left| \frac{s^2}{\omega_{np1}^2} + \frac{2z_{p1}}{\omega_{np1}} j\omega + 1 \right| - \dots$$

Note: Advantages to using the Bode magnitude plot

- 1) By using logarithms, the task of graphing the magnitude plot can be broken into pieces. The task is really one of learning how to graph each type of term shown above.
- 2) By using logarithms, a wide range of values for the magnitude of H can be displayed on one graph. A graph that displays a range of 100 dB for H_{dB} corresponds to a range 10^5 for H !
- 3) By plotting frequency with a logarithmic scale, one plot can give the behavior of H_{dB} over a wide range of frequency.

When sketching Bode plots by hand, one usually starts with the straight-line Bode plot, which is an approximation to the actual curve. Then one adjusts the straight-line Bode plot where easy and convenient. If one is neat, Bode magnitude plots can often be plotted to within ± 1 dB. For complicated curves or if higher accuracy is required, computer tools are used.

The figure below shows the straight-line plots for the individual Bode factors given on the previous page.

Terms in Straight-line Bode Magnitude Plots

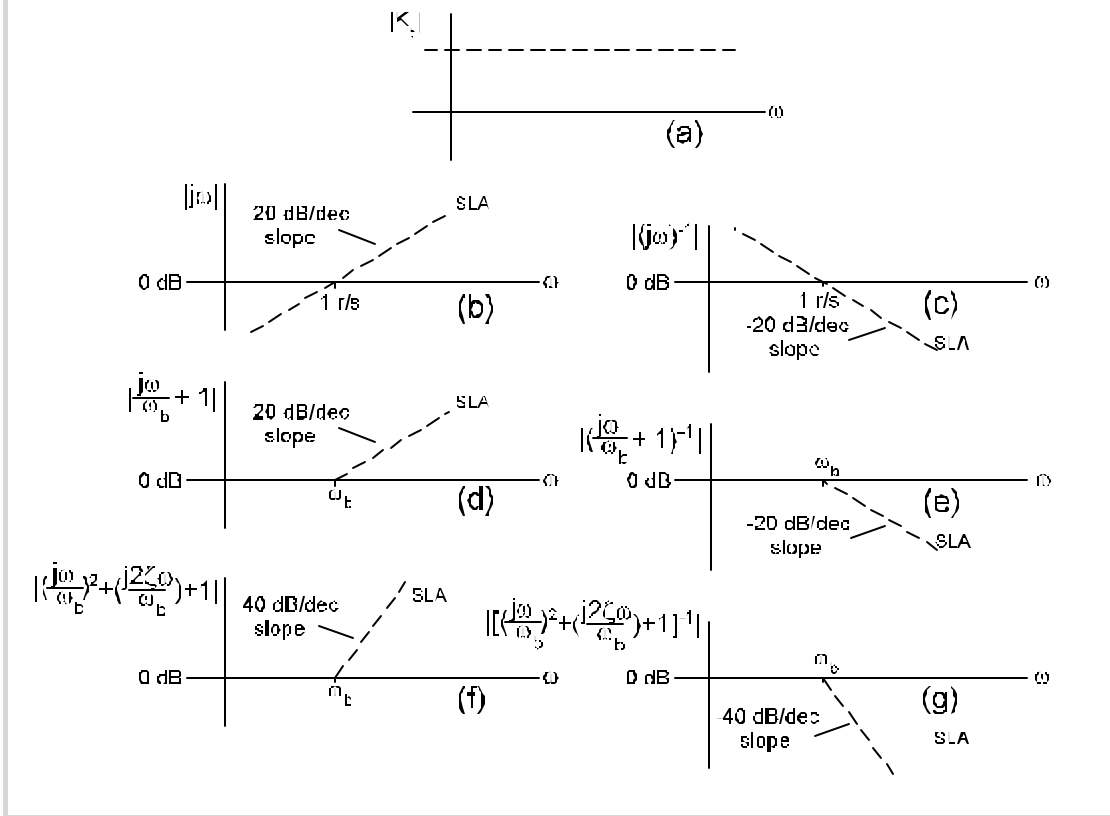


Figure 6.12: Straight Line Approximations for Terms in Bode Magnitude Plots

Note:

- 1) For the terms shown in (a), (b), and (c), the straight line plot is no different than the exact curve.
- 2) For the term in (d), the actual curve is 3db above the straight-line plot the break frequency ($\omega = \omega_b$) and 1 dB above the straight-line plot at the octave points ($\omega = \frac{1}{2}\omega_b$ and $\omega = 2\omega_b$).
- 3) For the term in (e), the actual curve is 3db below the straight-line plot the break frequency ($\omega = \omega_b$) and 1 dB below the straight-line plot at the octave points ($\omega = \frac{1}{2}\omega_b$ and $\omega = 2\omega_b$).
- 4) For the term in (f), the actual curve passes through $20 \log_{10} 2\zeta$ at the break frequency.
- 5) For the term in (g), the actual curve passes through $20 \log_{10} (2\zeta)^{-1}$ at the break frequency.
- 6) For the terms in (f) and (g), actual curves are plotted in Figure 6.13.

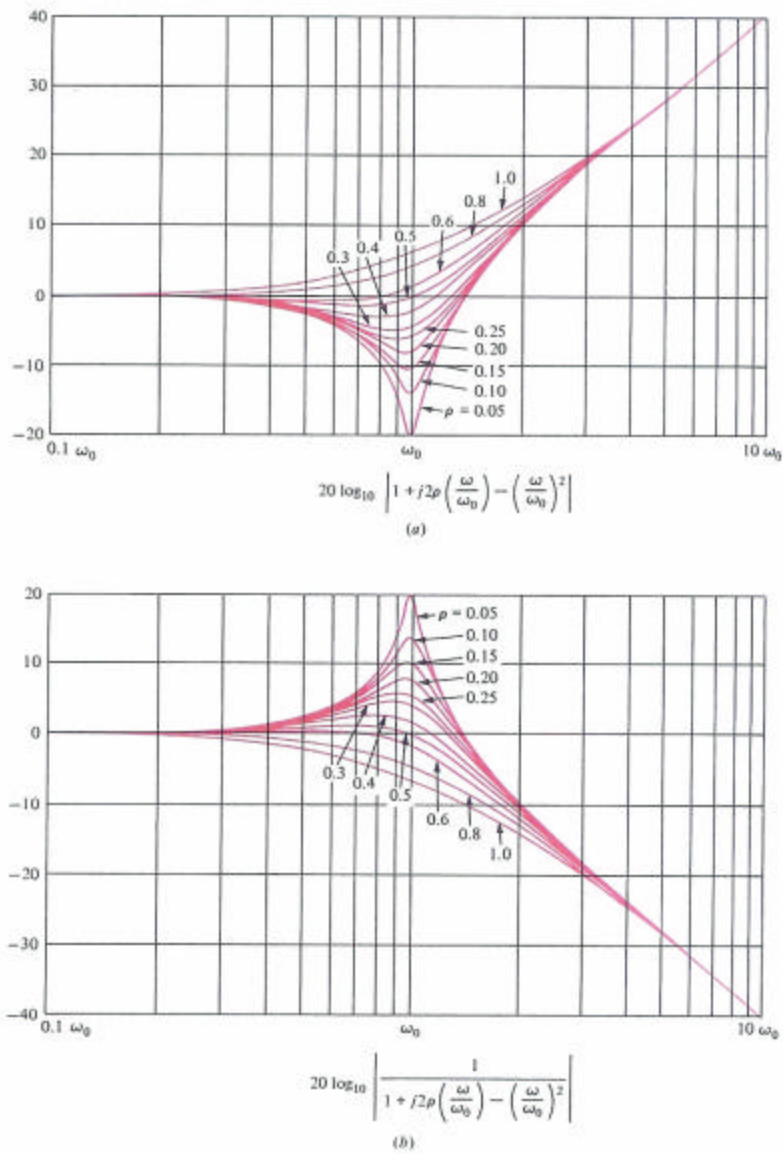


Figure 6.13 Bode magnitude plots for 2nd-order terms with various z

Example 6.7

Give the Bode magnitude plot for the low-pass transfer function shown below.

$$H(s) = \frac{2000}{s + 100} = \frac{20}{\frac{s}{100} + 1}$$

The magnitude plots for the individual Bode factors are shown in Fig. 6.14.

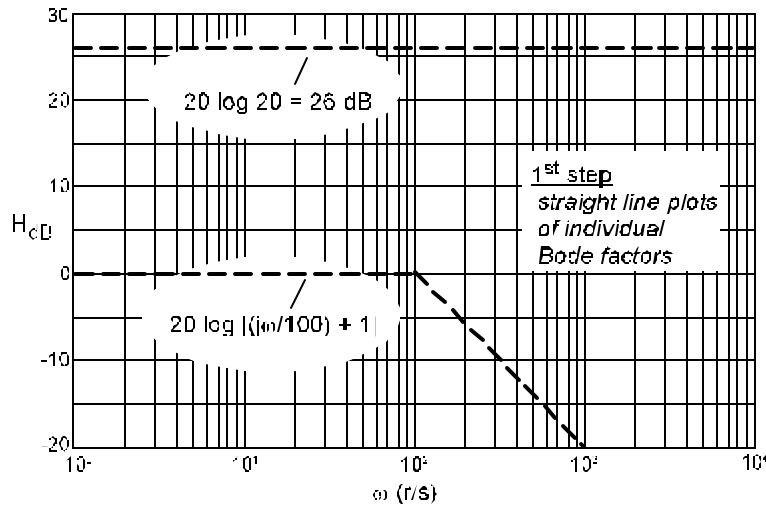


Figure 6.14

Add factors together to produce the composite straight line plot and adjust at $\frac{1}{2}\omega_b$ (-1db), ω_b (-3dB), and $2\omega_b$ (-1dB). The result is the actual curve.

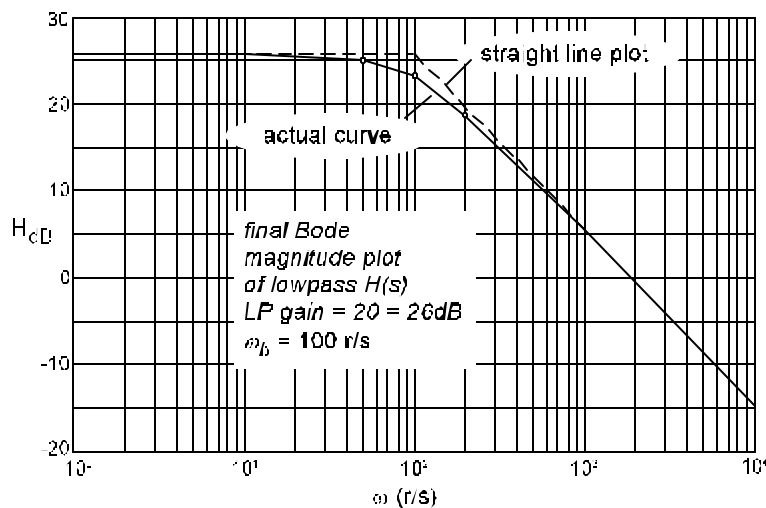


Figure 6.15

Example 6.8

Give the Bode magnitude plot for the bandpass transfer function shown below.

$$H(s) = \frac{2(10^4)s}{s^2 + 1010s + 10^4} = \frac{2s}{\left(\frac{s}{10} + 1\right)\left(\frac{s}{1000} + 1\right)}$$

The magnitude plots for the individual Bode factors are shown in Fig. 6.16.

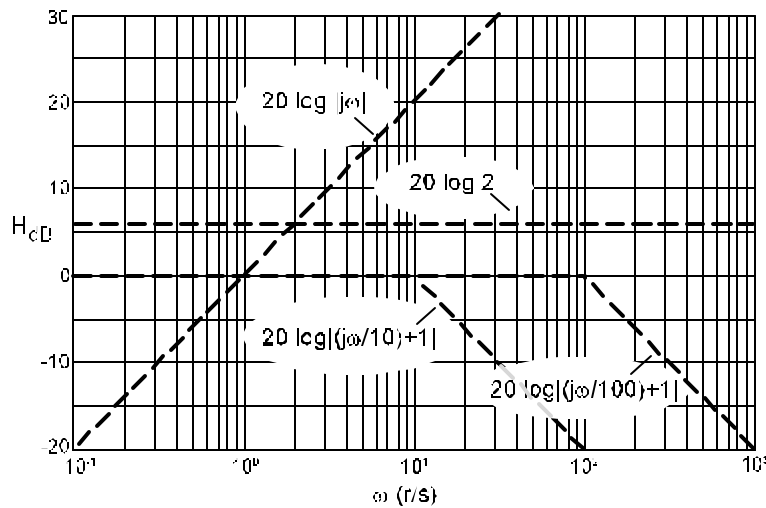


Figure 6.16

The composite straight-line plot shown in Fig. 6.17 is readily found by summing the individual terms. A convenient point with which to start is at $\omega = 1$ r/s, where all the terms are 0 dB except for the Bode constant term.

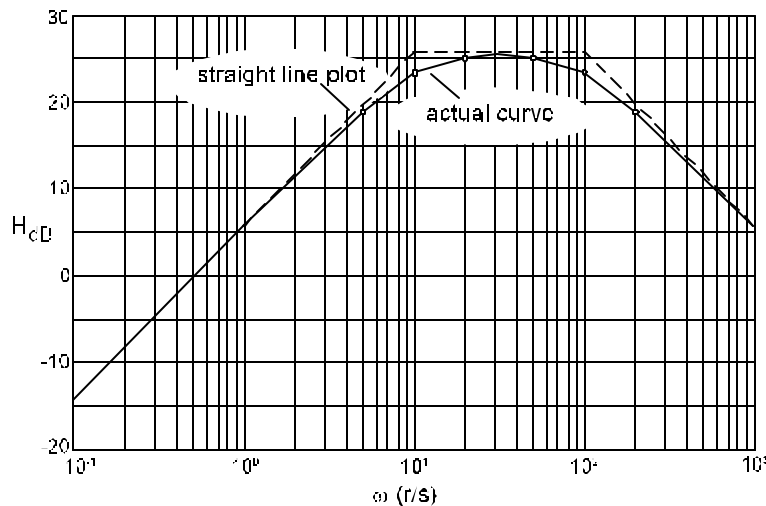


Figure 6.17

Bode Phase Plots

The phase response can naturally be expressed as a sum of the phases of the individual Bode terms. This is in contrast to the magnitude plot in which it appears only as a sum after taking the logarithm.

$$\begin{aligned}
 \mathbf{H} &= H \angle \phi_H \\
 \phi_H &= \angle K_b + (n-m) \angle(j\omega) + \angle\left(\frac{j\omega}{\omega_{bz1}} + 1\right) + \dots \\
 &+ \angle\left(\frac{(j\omega)^2}{\omega_{nz1}^2} + \frac{2z_{z1}}{\omega_{nz1}} j\omega + 1\right) + \dots - \angle\left(\frac{j\omega}{\omega_{bp1}} + 1\right) - \dots \\
 &- \angle\left(\frac{s^2}{\omega_{np1}^2} + \frac{2z_{p1}}{\omega_{np1}} j\omega + 1\right) - \dots
 \end{aligned}$$

As with magnitude plots, when sketching Bode phase plots by hand, one usually starts with the straight-line plot, which is an approximation to the actual curve. Then one adjusts the straight-line Bode plot where easy and convenient. For phase plots, the necessary adjustments are reasonably straightforward for 1st-order terms.

The adjustments for 2nd-order terms, particularly for terms with low damping ($\zeta \ll 1$), are not as simple and are more significant. See Fig. 6.19 for details.

As an alternative to using these figures, one can either calculate the actual value of phase at several frequencies within a decade of the break frequency to determine the degree (no pun intended) to which straight-line plot deviates from the actual curve, or, if computer tools are available, one would probably choose to use them.

Terms in Straight-line Bode Phase Plots

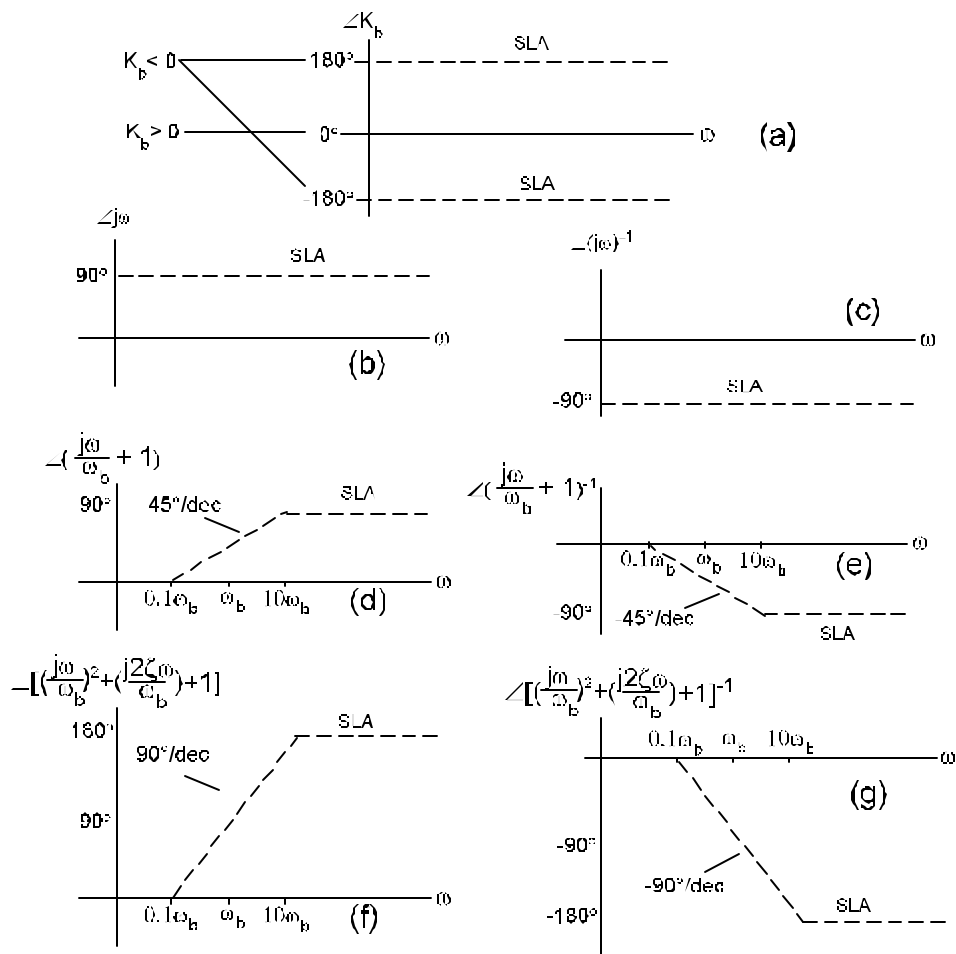


Figure 6.18 Straight Line Approximations to Terms in Bode Phase Plots

Note:

- 1) For the terms shown in (a), (b), and (c), the straight line plot is no different than the exact curve.
- 2) Note the multiple possible values the phase of $K_b < 0$. This is a particular instance of the general property of the phase being multivalued (for example, -270° is the same as $+90^\circ$). By convention, the phase of $H(s)$ is displayed as being between -180° and $+180^\circ$. For $K_b < 0$, its phase can be taken as either $+180^\circ$ or -180° whichever is the one which meets the convention.
- 3) For the term in (d), the actual curve is 5.7° above the straight-line plot at $\omega = 0.1\omega_b$ and 5.7° below the straight-line plot at $\omega = 10\omega_b$. The straight-line plot crosses the actual curve at $\omega = \omega_b$.

- 4) For the term in (e), the actual curve is 5.7° below the straight-line plot at $\omega = 0.1\omega_b$ and 5.7° above the straight-line plot at $\omega = 10\omega_b$. The straight-line plot crosses the actual curve at $\omega = \omega_b$.
- 5) For the term in (f) and (g), the straight-line plot crosses the actual curve at $\omega = \omega_b$. The degree of deviation away from the break frequency depends on ζ , the damping ratio, as illustrated in Fig. 6.19.

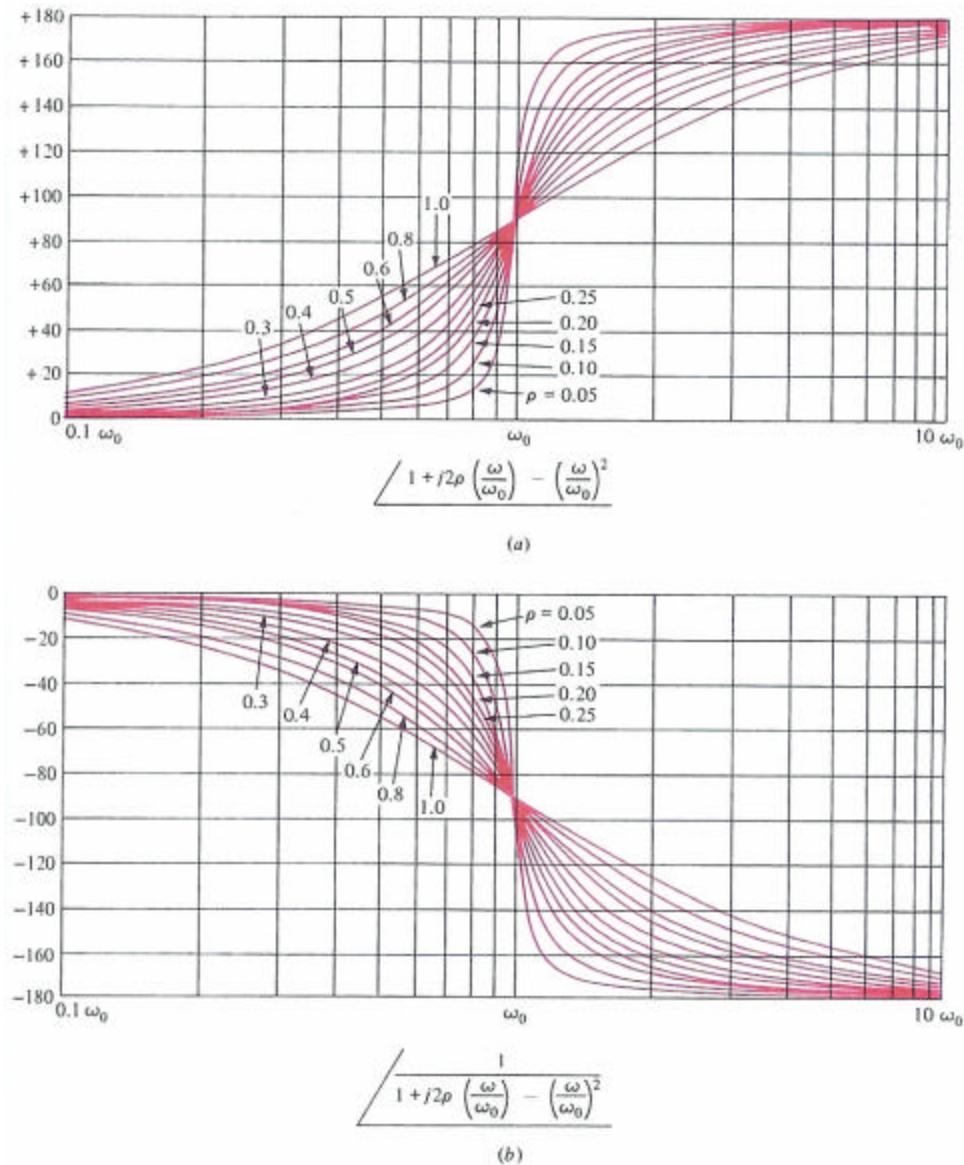


Figure 6.19 Bode phase plots for 2nd-order terms with various ζ

Example 6.9

Give the Bode phase plot for the bandpass transfer function shown below.

$$H(s) = \frac{2000s}{s + 100} = \frac{20s}{\left(\frac{s}{100} + 1\right)}$$

The phase plots for the individual Bode factors are shown in Fig. 6.18.

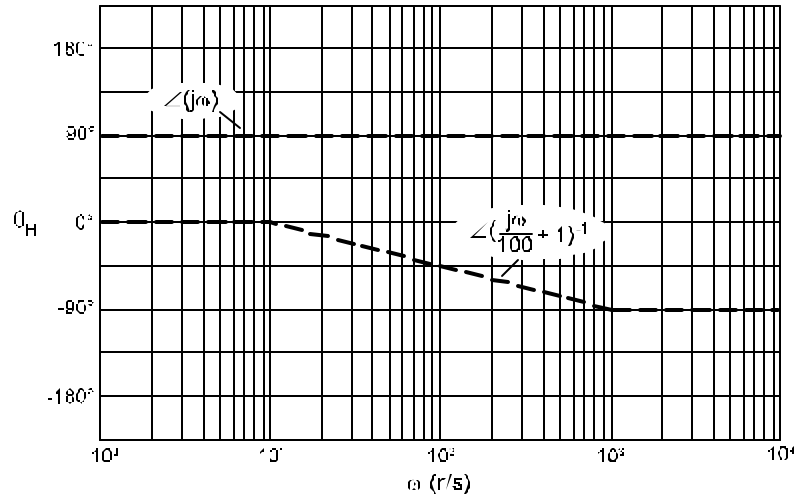


Figure 6.20

Add factors together to produce the composite straight line plot and adjust at $0.1\omega_b$ (-5.7°) and at $10\omega_b$ (5.7°). The result is the actual curve.

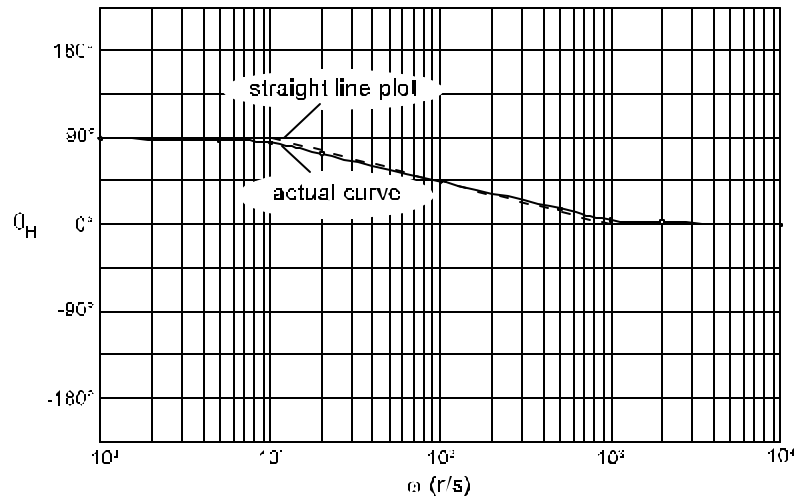


Figure 6.21

6.4 Fourier Series

Periodic waveforms can be expressed in as a sum of sinusoids, a Fourier series. Consider a periodic waveform, $x(t)$, with a period T_0 . The Fourier series for $x(t)$ appears below.

$$x(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t + \theta_n) \quad n = 1, 2, 3, \dots$$

$$\text{where } \omega_0 = \frac{2\pi}{T_0}$$

A_0 is the average value of $x(t)$ and allows Fourier series for periodic signals that have a non-zero average values.

$$A_0 = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} x(t) dt$$

Once all the A_n and θ_n are determined, the spectral content of $x(t)$ is known. That is, a complete description of $x(t)$ can be plotted in the frequency domain.

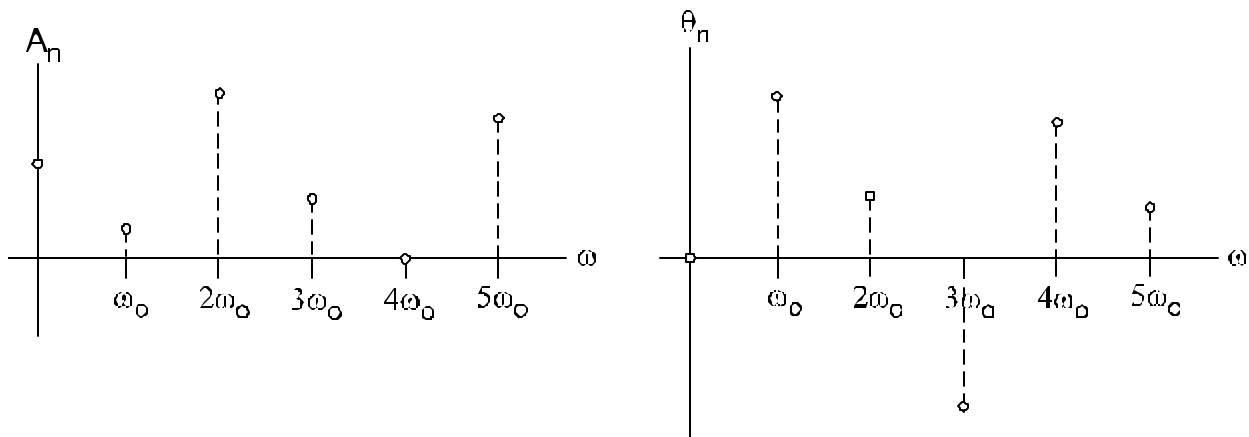


Figure 6.18 Fourier Magnitude and Phase Plots

The signal spectra, together with a system's frequency response, can be used to quickly determine the sinusoidal steady-state response to a periodic input. It should be reinforced at this point that, since many of these techniques rely upon superposition, *they are valid only for linear systems.*

Example 6.10

Suppose a system has a frequency response as shown below.

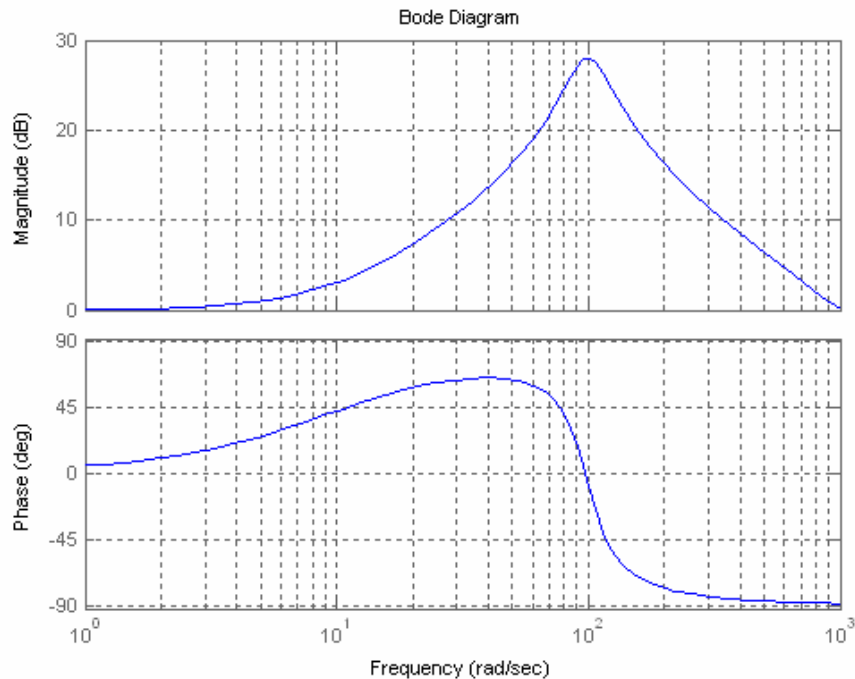


Figure 6.19 System Frequency Response, $H(j\omega)$

Recall a system's frequency response gives the sinusoidal steady-state response of the system. In particular, it shows, as a function of frequency, the ratio of output to input sinusoid amplitudes and the phase shift between output and input sinusoids.

In this example, the system response, magnitude and phase, is given via the Bode magnitude and phase plots shown in Fig. 6.19. Consider a periodic signal, $x(t)$, as an input into the system. What is the output?

The first step is to find the signal's Fourier series representation. The signal's spectra are shown in Fig. 6.20.

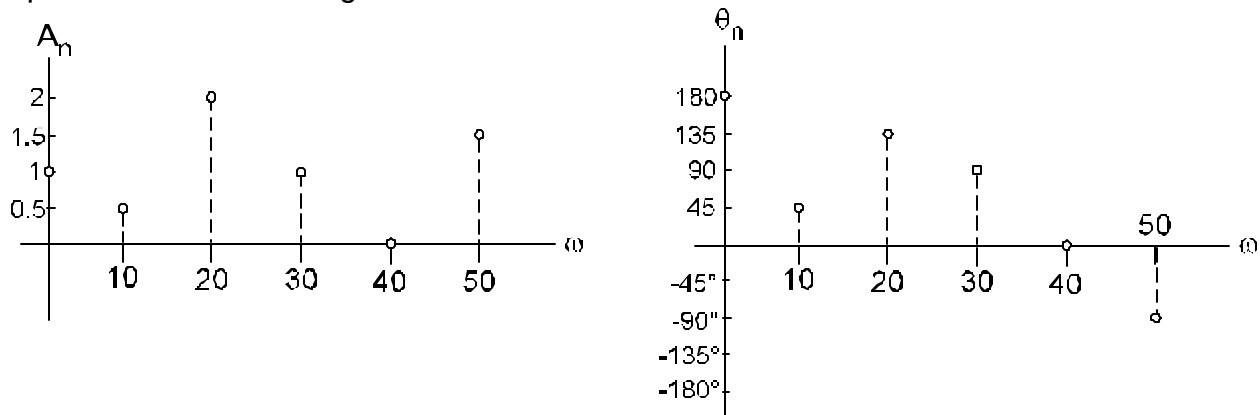


Figure 6.20 Fourier Series for $x(t)$

As one can see from the figure, $x(t)$, in terms of sinusoids is

$$x(t) = -1 + 0.5 \cos(10t + 45^\circ) + 2 \cos(20t + 135^\circ) + \cos(30t + 90^\circ) + 1.5 \cos(50t - 45^\circ)$$

To find the output, $y(t)$, one employs superposition and systematically applies the system response to the different frequency components of $x(t)$. Evaluating $H(j\omega)$ at $\omega = 10, 20, 30,$ and 50 r/s, one finds

$$H(j10) \cong 10^{\frac{3}{20}} \angle 43^\circ \cong 1.41 \angle 43^\circ$$

$$H(j20) \cong 10^{\frac{7.5}{20}} \angle 60^\circ \cong 2.37 \angle 60^\circ$$

$$H(j30) \cong 10^{\frac{11}{20}} \angle 63^\circ \cong 3.55 \angle 63^\circ$$

$$H(j40) \cong 10^{\frac{16}{20}} \angle 60^\circ \cong 6.31 \angle 60^\circ$$

To evaluate the DC response, we must make some assumptions if, indeed, we only know the system response for the frequencies included in Fig. 20. Let us assume that the DC response is $1 \angle 0^\circ$ which could be a reasonable extension from the available information in Fig. 6.21.

$$y(t) = -1(1) + 0.5(1.41) \cos(10t + 45^\circ + 43^\circ) + 2(2.37) \cos(20t + 135^\circ + 60^\circ) \\ + 1(3.55) \cos(30t + 90^\circ + 63^\circ) + 1.5(6.31) \cos(50t - 45^\circ + 60^\circ)$$

Simplifying, one obtains

$$y(t) = -1 + 0.705 \cos(10t + 88^\circ) + 4.74 \cos(20t - 165^\circ) \\ + 3.55 \cos(30t + 153^\circ) + 9.465 \cos(50t + 15^\circ)$$

This is the procedures used in using the techniques of Fourier series in sinusoidal steady-state analysis.

There are two observations in particular to be added. First the system response at $w = 0$, can only have a phase of $0^\circ, 180^\circ,$ or -180° . That is, the DC response must either be positive or negative. Second, the phase convention is to use phases between -180° and 180° . For example, when the output phase at $\omega = 30$ r/s was calculated to be 195° , the equivalent phase of -165° is used.

Finding Fourier Series

Given $x(t)$, some means of finding the Fourier coefficients is needed. A_0 represents the average value of $x(t)$.

$$A_0 = \frac{1}{T_0} \int_{t_0}^{t_0 + T_0} x(t) dt$$

The other Fourier coefficients, A_n and θ_n for $n = 1, 2, 3, \dots$, can readily be determined. Using Euler's Identity,

$$\begin{aligned} x(t) &= A_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t + \mathbf{q}_n) = A_0 + \sum_{n=1}^{\infty} \frac{A_n}{2} [e^{j(n\omega_0 t + \mathbf{q}_n)} + e^{-j(n\omega_0 t + \mathbf{q}_n)}] \\ &= A_0 + \sum_{n=1}^{\infty} \left[\left(\frac{A_n}{2} \angle \mathbf{q}_n \right) e^{jn\omega_0 t} + \left(\frac{A_n}{2} \angle -\mathbf{q}_n \right) e^{-jn\omega_0 t} \right] \\ &= A_0 + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \mathbf{X}_n e^{jn\omega_0 t} \\ &= \sum_{n=-\infty}^{\infty} \mathbf{X}_n e^{jn\omega_0 t} \end{aligned}$$

This form is called the complex Fourier series. The advantage to the complex Fourier series is that the \mathbf{X}_n can be readily found from

$$\mathbf{X}_n = \frac{1}{T_0} \int_{t_0}^{t_0 + T_0} x(t) e^{-jn\omega_0 t} dt, \quad \text{where } \mathbf{X}_n = X_n \angle \mathbf{q}_n$$

When one studies the development above, it is clear that $\mathbf{X}_n = \mathbf{X}_{-n}^*$. It is also clear that \mathbf{X}_0 is the average value of $x(t)$ and that $\mathbf{X}_0 = A_0$. That is, \mathbf{X}_0 is either a positive or a negative real number.

Once the \mathbf{X}_n have been determined, the A_n and θ_n are known by inspection.

$$\begin{aligned} A_0 &= X_0 \\ A_n &= 2X_n \quad n = 1, 2, 3, \dots \\ \mathbf{q}_n &= \mathbf{q}_n \quad n = 1, 2, 3, \dots \end{aligned}$$

Example 6.11

Find the Fourier series for the signal shown below.

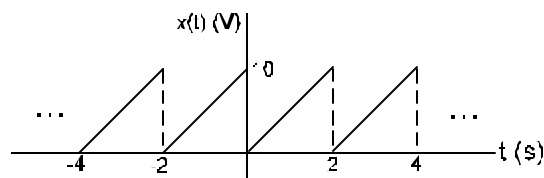


Figure 6.21

Since the waveform is periodic with a period of $T_o = 2s$, it can be represented as a Fourier Series.

$$x(t) = A_o + \sum_{n=1}^{\infty} A_n \cos(n\omega_o t + q_n) = A_o + \sum_{n=1}^{\infty} A_n \cos(np t + q_n) \quad n = 1, 2, 3, \dots$$

$$\text{where } \omega_o = \frac{2p}{T_o} = \frac{2p}{2s} = p \text{ r/s}$$

$$\text{where } A_o = X_o = \frac{1}{T_o} \int_{t_o}^{t_o + T_o} x(t) dt \quad n = 0$$

$$\frac{A_n}{2} \angle q_n = X_n = \frac{1}{T_o} \int_{t_o}^{t_o + T_o} x(t) e^{-jn\omega_o t} dt \quad n \neq 0$$

Since $x(t)$ is periodic (after all, it must be to have a Fourier series), evaluating the above integrals over *any* period will give identical results. To evaluate the integrals, we might as well choose the most convenient period which, in this case is for $0 < t < 2s$. In this range, $x(t)$ is described as

$$x(t) = 5t \text{ V} \quad \text{for } 0 < t < 2s$$

The average value can be found first

$$A_o = \frac{1}{2} \int_0^{2s} 5t dt = \frac{15}{2} \frac{t^2}{2} \Big|_{t=0}^{t=2s} = \frac{15}{2} \frac{4}{2} \text{ V} = 5 \text{ V}$$

The A_n and θ_n come next

$$\frac{A_n}{2} \angle q_n = X_n = \frac{1}{2} \int_{t=0}^{2s} 5t e^{-jnpt} dt \quad n = 1, 2, 3, \dots$$

$$\begin{aligned} \frac{A_n}{2} \angle q_n &= \frac{5}{2} \int_{t=0}^{2s} t e^{-jnpt} dt = \frac{5}{2} t \left(\frac{1}{-jn p} \right) e^{-jn p t} \Big|_0^{2s} - \left(\frac{1}{-jn p} \right) \int_0^{2s} e^{-jn p t} dt \\ &= j \frac{5}{n p} e^{-jn 2p} + \frac{1}{(n p)^2} (e^{-jn 2p} - 1) = j \frac{5}{n p} e^{-jn 2p} \\ &= e^{j \frac{p}{2}} \frac{5}{n p} e^{-jn 2p} = \frac{5}{n p} e^{j(\frac{p}{2} - n 2p)} = \frac{5}{n p} e^{j \frac{p}{2}} \end{aligned}$$

$$A_n = \frac{10}{n p} \quad n = 1, 2, 3, \dots$$

$$q_n = \frac{p}{2}$$

The series is shown below.

$$x(t) = 5 + \frac{10}{p} \cos(pt + 90^\circ) + \frac{5}{p} \cos(2pt + 90^\circ) + \frac{10}{3p} \cos(3pt + 90^\circ) + \frac{5}{2p} \cos(4pt + 90^\circ) + \dots$$
$$x(t) = 5 - \frac{10}{p} \sin(pt) - \frac{5}{p} \sin(2pt) - \frac{10}{3p} \sin(3pt) - \frac{5}{2p} \sin(4pt) + \dots$$

6.5 Summary

Frequency domain analysis is important for any engineer to know. It is not just an electrically engineering tool. Mechanical engineers, physicists, biomedical engineers, chemical engineers, mathematicians civil engineers, and computer engineers all find use for frequency domain analysis. Frequency domain analysis is based on phasor analysis. To truly understand frequency domain analysis, one should first become familiar with phasor techniques.

It is important to bear in mind that frequency domain techniques are designed to find the sinusoidal steady-state response of systems. The natural response is not found when one uses frequency domain analysis. The range of application for frequency domain analysis is greatly extended using Fourier techniques. Using Fourier series, for example, allows the steady-state response of periodic inputs, not just sinusoidal inputs, to be found. (Using the more advanced techniques of *Fourier Transforms*, one can find the response even to non-periodic inputs.) It is important to realize that many Fourier techniques depend upon linearity and are not applicable to nonlinear systems. For instance, design example 6.7.1 below depends fundamentally on the linearity of the system. This type of analysis would not be valid if the system were nonlinear.

Once a fundamental understanding of phasor technique is achieved, many find s-domain analysis a convenient means of reducing the amount of complex arithmetic when working the frequency domain analysis. The general outline of the procedure, starting from the differential equations which govern system behavior in the time domain, is to obtain the Laplace transform of the system (in frequency domain analysis the initial conditions are taken as zero) and then substitute $j\omega$ for s to obtain the complex frequency domain relation between input and output.

Fourier Series provide a method of splitting periodic signals into their sinusoidal components. It is most useful to think of periodic signals as being described by their cosine series since this form of the series provides one with the signals spectra—that is, with the magnitudes and phases of its sinusoidal components. This form, with its associated spectra, is readily used in conjunction with a systems frequency response, often in terms of its Bode magnitude and phase plots.

6.6 Computer Tools and Other Resources

MATLAB is a convenient tool to generate Bode plots. Consider the transfer function shown below.

$$H(s) = \frac{s + 100}{s^2 + 40s + 10000}$$

The MATLAB code for the above transfer function is

```
clear;  
s=tf('s');  
TF=(s+100)/(s^2 + 40*s + 10000);  
bode(TF)  
grid
```

A screen shot of the plot is shown to the right in Fig. 6.22a. By copying and pasting, one can obtain the figure shown in Fig. 6.22b.

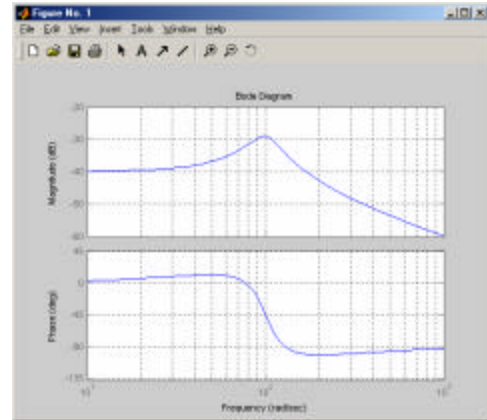


Figure 6.22a

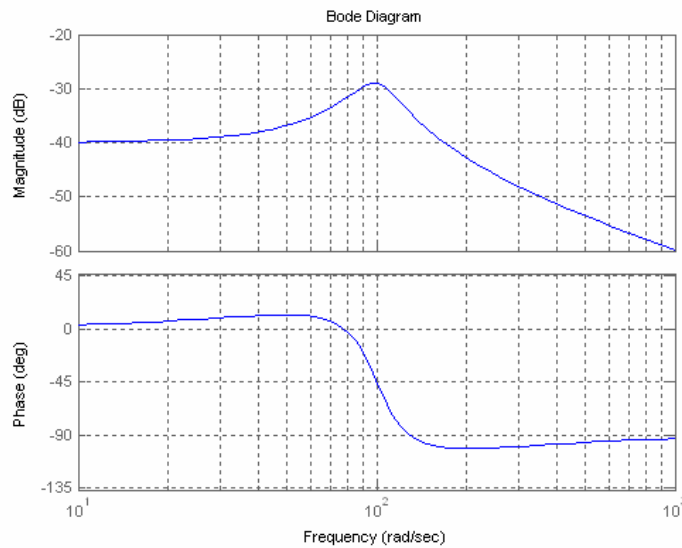


Figure 6.22b

MAPLE is a convenient tool to calculate Fourier series. Consider the time-domain response of a second-order system to a train of periodic pulses as shown in Fig. 6.23.

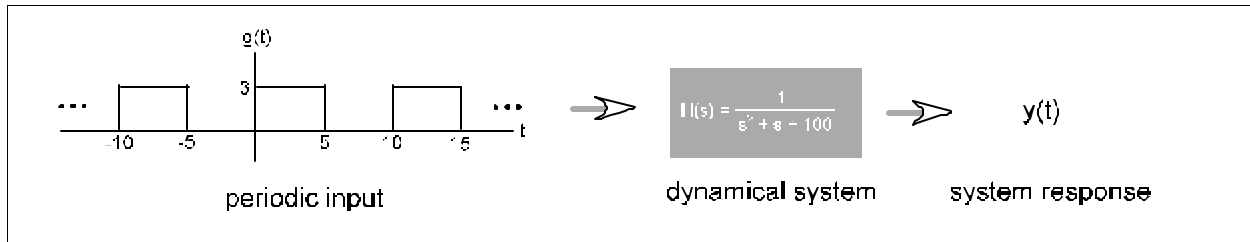


Figure 6.23

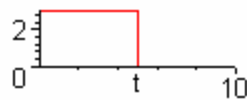
The required MAPLE code is shown below.

Pulse train amplitude = 3 with a period of 10.

```
> with(plots):T:=10:
> w0:=2*Pi/T:
```

Define the function.

```
> g := piecewise(t<5,3,t<10,0 ):
> plot(g,t=0..T);
```



Calculate the Fourier Coefficients

```
> Xn:=n->1/T*int(g*exp(-I*2*Pi*n*t/T),t=0..T):
> X_mag0:=abs(Xn(0)):
> X_mag:=n->2*abs(Xn(n)):
> X_angle:=n->argument(Xn(n)):
```

Converting phase into degrees

```
> X_degrees:=n->180*argument(Xn(n))/Pi:
```

Defining the Fourier expansion

```
> f:=m->X_mag0+sum(X_mag(k)*cos(k*w0*t+X_angle(k)),k=1..m):
```

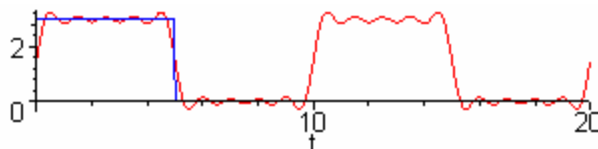
Evaluating to $n=5$. Only sine terms appear (or cosines with $\pm 90^\circ$ phases) due to the fact that $g(t)$ is odd about its average value. Also notice that the even harmonics are missing due to odd half-wave symmetry.

```
> evalf(f(5),5);
```

$$1.5000+1.9099*\sin(.62832*t)+.63662*\sin(1.8850*t)+.38197*\sin(3.1416*t)$$

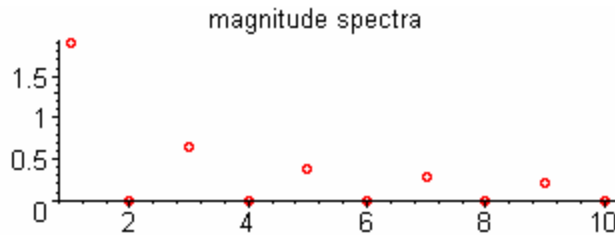
Plot input and output

```
> plot([g,f(10)],t=0..2*T,color=[blue,red]);
```

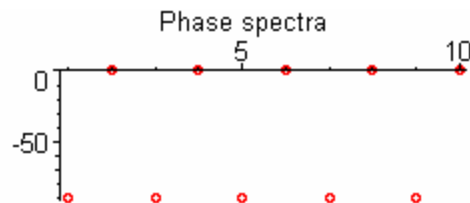


Plot magnitude and phase of the Fourier Coefficients

```
> mag_points:=evalf({seq([i,X_mag( i )], i = 1..10 ))):
> plot(mag_points,style=point,symbol=circle,title="magnitude spectra");
```



```
> phase_points:=evalf({seq([i,X_degrees( i )], i = 1..10 ))):
> plot(phase_points,style=point,symbol=circle,title="Phase spectra");
```



Suppose $g(t)$ is used as an input to a system defined by transfer function, TF below.

```
> TF:=1/(s^2+s+100):
```

Find the magnitude and phase after substituting in $l \cdot \omega$ ($l = j = \sqrt{-1}$)

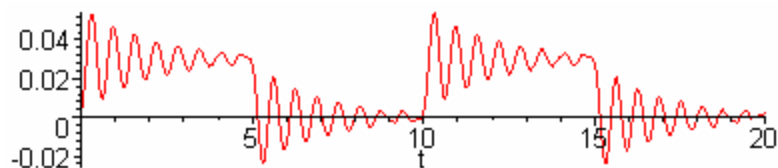
```
> magTF:=omega->abs(subs(s=l*omega,TF)):
> angTF:=omega->argument(subs(s=l*omega,TF)):
```

Now find the time response

```
> y:=m->X_mag0*magTF(0)+sum(X_mag(k)*magTF(k*w0)*cos(k*w0*t+X_angle(k)+angTF(k*w0)),k=1..m):
> evalf(y(5),5):
```

Plot the result

```
> plot(y(25),t=0..2*T);
```



6.7 Design Examples

The list of applications where frequency-domain analysis is used is nearly inexhaustible. Analyzing the loading of bridges, electronic circuit design, economic analysis, printed circuit board layout, vibration analysis, and optical communications is a small sample of the applications. This lists hints at the wide variety of applications in which frequency-domain analysis is used. Engineers and scientists in a wide variety of disciplines use frequency-domain techniques to complement time-domain techniques when analyzing dynamical systems.

6.7.1. Design Example 1: Active Filters

Filter design is one area in which frequency-domain analysis is particularly useful and one which is particularly useful in instrumentation systems. Filters are frequency selective circuits. Think of an equalizer on an audio system. The equalizer is a series of filters that can adjust the strength of various frequency bands in order to make the music sound better.

Filters can be separated into two categories. Passive filters are filters which used only passive components such a resistors, inductors, or capacitors. In contrast, active filters are ones which use active components such as operational amplifiers. Active filters are widely used in instrumentation and measurement. Active filters have the advantage in that they can be designed to both amplify and filter.

Looking at the filter as an input/output relation, filters are classified by how the input and output magnitudes are related at different frequencies. In this example, three types will be considered: low pass, high pass, and bandpass.

Low Pass

Low pass filters pass low frequency signals from input to output, but attenuate higher frequency signals. The active low pass filter is able to amplify low frequency signals while, at the same time, attenuating higher frequency signals.

In this design example, the TL072 dual op-amp (from SGS-Thomson Microelectronics) will be used for illustration, and pin-outs will correspond to the TL072.

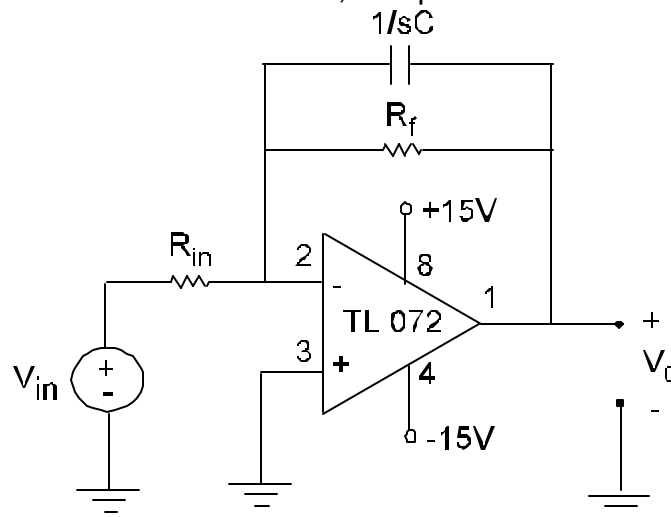


Figure 6.24 Active low pass filter

This op-amp is connected in the inverting configuration, and the transfer function is:

$$\frac{V_o}{V_{in}} = - \frac{Z_f}{Z_{in}} = - \frac{R_f}{R_{in}} \frac{1}{sR_f C + 1}$$

When the transfer function is expressed in this form, the low frequency gain and break frequency can be clearly seen.

$$\omega_b = \frac{1}{R_f C} \quad \left| \frac{V_o}{V_{in}} \right|_{\omega < \omega_b} \cong \frac{R_f}{R_{in}}$$

The Bode magnitude plot of the frequency response is

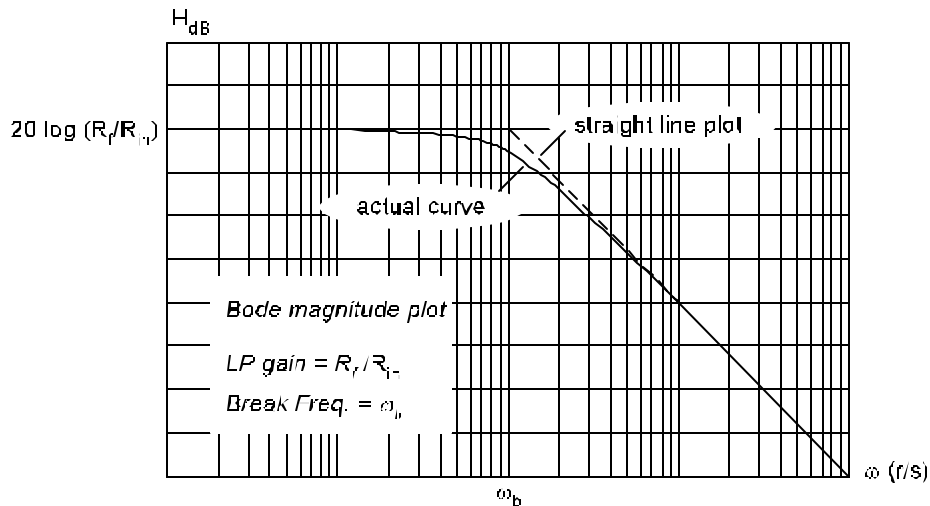


Figure 6.25

High pass

High pass filters pass high frequency signals from input to output, but attenuate higher frequency signals. The active high pass filter is able to amplify high frequency signals while, at the same time, attenuating lower frequency ones.

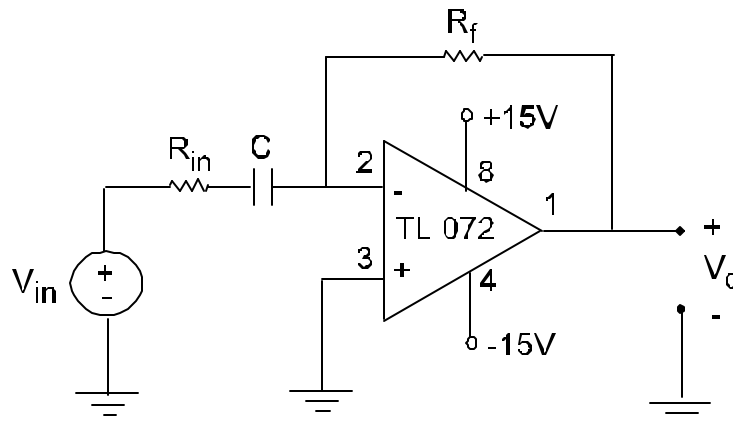


Figure 6.26 Active high pass filter

This op-amp is connected in the inverting configuration, and the transfer function is:

$$\frac{V_o}{V_{in}} = - \frac{Z_f}{Z_{in}} = - \frac{sR_f C}{sR_{in} C + 1}$$

When the transfer function is expressed in this form, the high frequency gain and break frequency can be clearly seen.

$$\omega_b = \frac{1}{R_{in} C} \quad \left| \frac{V_o}{V_{in}} \right|_{s \rightarrow \infty} \cong \frac{R_f}{R_{in}}$$

The Bode magnitude plot of the frequency response is

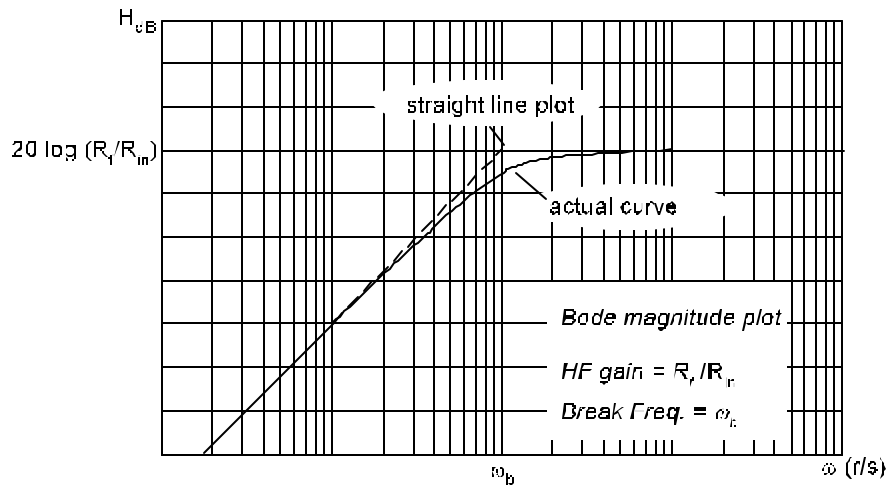


Figure 6.27

Bandpass

Bandpass filters are useful when one wishes to pass a band of frequencies while, at the same time, attenuating both lower and higher frequencies. Active bandpass filters are able to amplify signals within a band of frequencies while, at the same time, attenuating lower or higher frequencies (frequencies outside the “pass band”).

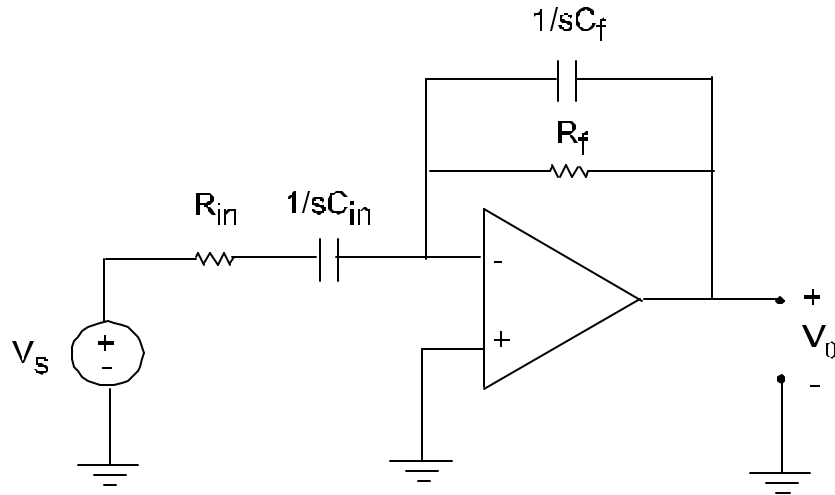


Figure 6.28 Active bandpass filter

General analysis

$$\frac{V_0}{V_s} = -\frac{Z_f}{Z_{in}} = -\frac{R_f / (sR_f C_f + 1)}{(sR_{in} C_{in} + 1) / sC_{in}} = -\frac{sR_f C_{in}}{s^2 R_f R_{in} C_f C_{in} + s(R_f C_f + R_{in} C_{in}) + 1}$$

The resonant frequency occurs within the pass band.

$$\omega_n = \sqrt{\frac{1}{R_f R_{in} C_f C_{in}}}$$

The gain at this frequency is

$$\left| \frac{V_0}{V_s} \right| = \frac{R_f C_{in}}{R_f C_f + R_{in} C_{in}}, \quad s = j\omega = j\omega_n$$

The element values for the bandpass filter can be chosen so that the bandpass filter can be seen as acting as a highpass filter for low frequencies, as an amplifier for pass band frequencies and as a low pass filter for high frequencies.

Note:

1. At low frequencies, the impedance of C_f is much larger than R_f . Replacing C_f by an open circuit, one can see the filter operates essentially as a high pass filter.
2. In the pass band, $\frac{1}{R_{in} C_{in}} < \omega < \frac{1}{R_f C_f}$, the impedance of C_f is still much larger than R_f plus the impedance of C_{in} is much smaller than R_{in} . Replacing C_f by an open circuit and C_{in} by a short circuit, one can see the filter functions essentially as an amplifier.

- At high frequencies, the impedance of C_{in} remains much smaller than R_{in} , while the impedance of C_f can no longer be considered much larger than R_f . Replacing C_{in} by a short circuit, one can see the filter functions essentially as a low pass filter.

Showing filter behavior via a Bode plot,

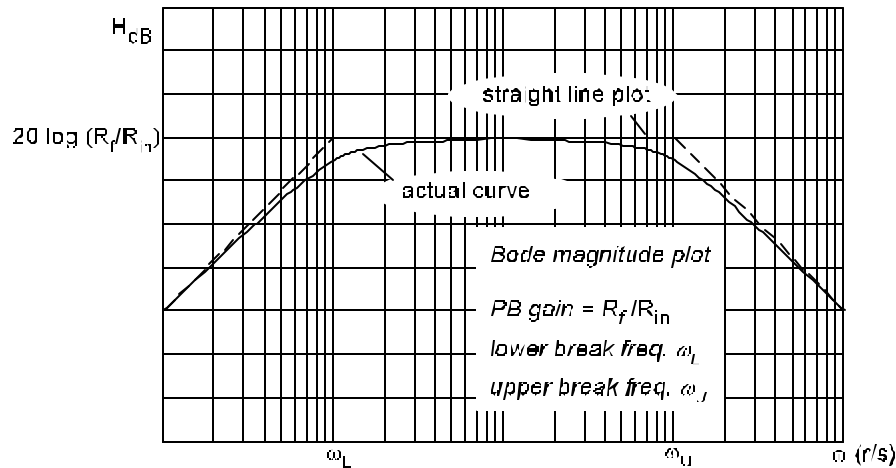


Figure 6.29 Bandpass filter with $R_{in}C_{in} \gg R_fC_f$

Note that the advantages in the bandpass filter designed as described above is that the gain can be made relatively constant over a wide range of frequencies while the lower and upper break frequencies can be readily adjusted independently of one another, so long as the condition $R_{in}C_{in} \gg R_fC_f$ is not violated.

6.7.2. Design Example 2: Dynamic Measurement

A thermocouple is a temperature transducer formed when two dissimilar metals are joined together. In this design example, a type N thermocouple (formed by Cr/Si/Ni and Si/Mg/Ni alloys) is used to measure the temperature of an exhaust gas which varies sinusoidally with time.

The thermocouple can be modeled as a first order system. How does the thermocouple's temperature varies as a function of time?

$$\text{time-domain dynamical equation} \quad t \frac{dT_{tc}}{dt} + T_{tc} = T_{gas}$$

$$\text{frequency-domain (phasor) equation} \quad j\omega t T_{tc} + T_{tc} = T_{gas}$$

From the frequency-domain equation, the frequency response of the system can be found. Note the input to the thermocouple is the gas temperature and the response (the output) is the thermocouple temperature.

$$\frac{T_{tc}}{T_{gas}} = \frac{1}{j\omega t + 1}$$

Note the frequency response has the form of a low-pass filter with a gain of one. This reflects the physics of the situation. If the gas temperature were a constant temperature (low frequency) the thermocouple would become equal to the gas temperature. The thermocouple temperature would not grow larger than the gas temperature, therefore the gain is one as is typical for situations involving heat transfer. If the temperature of the gas changed varied sinusoidally at a high frequency, the thermal mass of the thermocouple would prevent the thermocouple temperature from following the variations of gas temperature, but rather would remain at the average temperature.

If the temperature of the gas varied sinusoidally, sufficiently slowly so that the thermocouple temperature responds to the variations but too quickly for the thermocouple temperature to track it precisely. In this event the engineer or scientist would need to know about frequency response.

Consider that a given thermocouple with a time constant of 5 seconds is used to measure a gas temperature which varies as

$$T_{gas} = [300 + 50 \cos(0.1t)] \text{ } ^\circ\text{C}$$

$$\frac{T_{tc}}{T_{gas}} = \frac{1}{j\omega t + 1} = \frac{1}{j\omega 5 + 1}$$

Superposition must be used, and the average temperature of 300 °C (corresponding to an input with $\omega = 0$) considered separately from the variation at $\omega = 0.1$ r/s.

$$\text{at } \omega = 0, T_{tc} = [H(\omega=0)]300\angle 0^\circ \text{ } ^\circ\text{C} = 300 \text{ } ^\circ\text{C}$$

$$\text{at } \omega = 0.1, T_{tc} = [H(\omega=0.1)]50\angle 0^\circ \text{ } ^\circ\text{C} \cong 44.7\angle -26.6^\circ \text{ } ^\circ\text{C}$$

The thermocouple temperature varies with time as

$$T_{tc}(t) = [300 + 44.7 \cos (0.1t - 26.6^\circ)] \text{ } ^\circ\text{C}$$

As can be seen clearly, the thermocouple would indicate lower temperature swings than those actually present in the gas temperature and changes in the thermocouple temperature would lag the gas temperature by about 0.74 seconds.