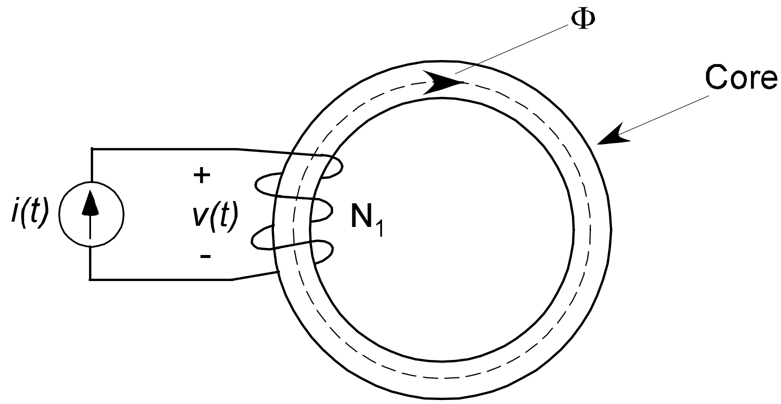
Review of Ideal Transformers

The ideal transformer model is based on a pair of mutually-coupled coils. When a variable magnetic field that is produced in one part of a circuit, links with components in another part of the circuit (or even with a different, but nearby circuit) a voltage is induced. This effect is called MUTUAL INDUCTANCE. It has some similarities with SELF INDUCTANCE, so we will have a brief review of this.

Consider a ring magnet with N_1 turns on the coil. The magnetic FLUX, Φ, flows around the ring and is analogous to current in an electrical circuit. The units of flux are webers (Wb).



The MAGNETOMOTIVE FORCE (MMF), $\mathscr{F}$, pushes the flux around and is analogous to voltage in an electrical circuit. The units of MMF are ampere-turns, and $\mathscr{F} = N_1 i$.

There is a quantity called RELUCTANCE, $\Re$, which is analogous to resistance, i.e.

$$\Re = \frac{\mathscr{F}}{\Phi} = \frac{N_1 i}{\Phi} \qquad \text{is similar to Ohm's Law}$$

The voltage induced in the coil is *v(t)*, which is given by Lenz's and Faraday's Laws as:

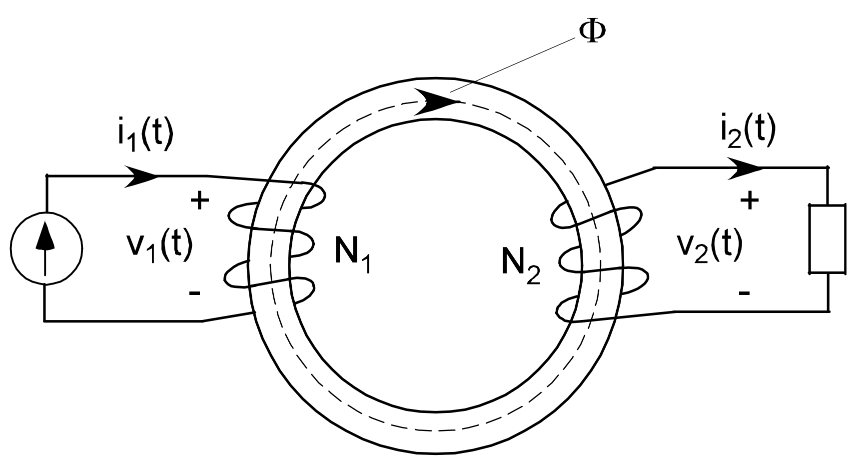
$$v(t) = L_1 \frac{di}{dt} = N_1 \frac{d\Phi}{dt}$$

Where: $L_1 = \frac{N_1 \Phi}{i}$ is the SELF INDUCTANCE of the coil and its units are henries (H).

Also: $L_1 = \frac{N_1^2}{\Re}$

The energy stored in the coil is: $\boxed{W_f = ½ L_1 i^2}$

If we now put a second coil (with N_2 turns) on the ring it will be linked by the same flux Φ, which will cause a second voltage to be induced.



The voltage induced in the coil is $v_2(t)$, which is given by:

$$v_2(t) = N_2 \frac{d\Phi}{dt} \qquad \text{and} \qquad v_1(t) = L_1 \frac{di_1}{dt} = N_1 \frac{d\Phi}{dt}$$

This means that: $\dfrac{d\Phi}{dt} = \dfrac{L_1}{N_1}\dfrac{di_1}{dt}$

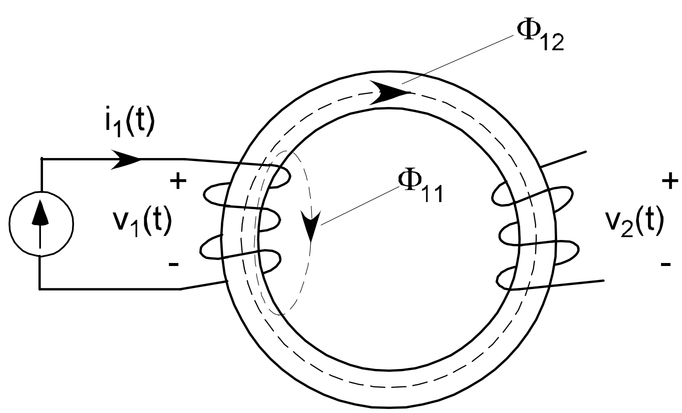
Which leads to: $v_2(t) = N_2 \dfrac{L_1}{N_1}\dfrac{di_1}{dt}$ and since, $L_1 = \dfrac{N_1^2}{\Re}$

$$v_2(t) = \frac{N_1 N_2}{\Re}\frac{di_1}{dt}$$

The term: $\dfrac{N_1 N_2}{\Re} = M_{ideal}$ is the "ideal" MUTUAL INDUCTANCE of the two coils. Like Self Inductance, its units are henries.

i.e. $\boxed{v_2(t) = M\dfrac{di_1}{dt}}$ is the voltage induced in coil 2 due to the current flowing in coil 1.

Since $L_1 = \dfrac{N_1^2}{\Re}$ and $L_2 = \dfrac{N_2^2}{\Re}$ then $M_{ideal} = \sqrt{L_1 L_2}$ if no flux leaks. In practice not all of the flux links both coils and some of it leaks, causing the actual M to be reduced.



$\Phi = \Phi_{11} + \Phi_{12}$ is the total flux produced by the MMF in coil 1, i.e.
$\Phi = N_1 i_1(t)$
Φ_{12} is the "main flux"
Φ_{11} is the "leakage flux"

For non-ideal coupling between two coils: $\boxed{M = k\sqrt{L_1 L_2}}$

Where "k" is the coefficient of coupling and can be thought of as the fraction of main flux to total flux. Obviously, $0 < k < 1.0$.

Mutual Inductance is the basis for transformer action in power applications, it is what enables a "search coil" to work in communication applications, and it causes much of the interference experienced in instrumentation applications.

When one coil is open-circuited it experiences an induced voltage due to the change in current in the other coil, i.e. the mutually induced voltages when one coil is open-circuited are:

$v_1(t) = M\dfrac{di_2}{dt}$ when coil 1 is O/C, and $v_2(t) = M\dfrac{di_1}{dt}$ when coil 2 is O/C.

The total energy stored in the coils is: $\boxed{W_f = \tfrac{1}{2}L_1 i_1^2 + \tfrac{1}{2}L_2 i_2^2 + M i_1 i_2}$ where i_1 and i_2 are the instantaneous values of the currents at the time of interest.

Example

A coil with a self inductance of 150 mH is energized with a current of $i_1(t) = 5\sin 500t$ A. This causes a second coil to develop an open-circuit voltage of $v_2(t) = 389.7\cos 500t$ V. If 10% of the developed flux leaks, determine the mutual inductance of the pair, the self inductance of coil 2 and the maximum stored energy.

We know: $$v_2(t) = 389.7\cos 500t = M\frac{di_1}{dt}$$

Also: $$\frac{di_1}{dt} = 2500\cos 500t \quad \text{A/s}$$

$$\therefore \quad M = v_2(t) \Big/ \frac{di_1}{dt} = \frac{389.7\cos 500t}{2500\cos 500t} \quad = \quad \boxed{155.9\text{ mH .}}$$

Since: $$M = k\sqrt{L_1 L_2} \qquad \text{then} \quad L_2 = \frac{M^2}{k^2 L_1}$$

And since 10% of the flux leaks, $k = 0.9$

$$\therefore \quad L_2 = \frac{0.1559^2}{0.9^2 \times 0.15} \quad = \quad \boxed{200\text{ mH .}}$$

We know: $$W_f = \tfrac{1}{2}L_1 i_1^2 + \tfrac{1}{2}L_2 i_2^2 + M i_1 i_2$$

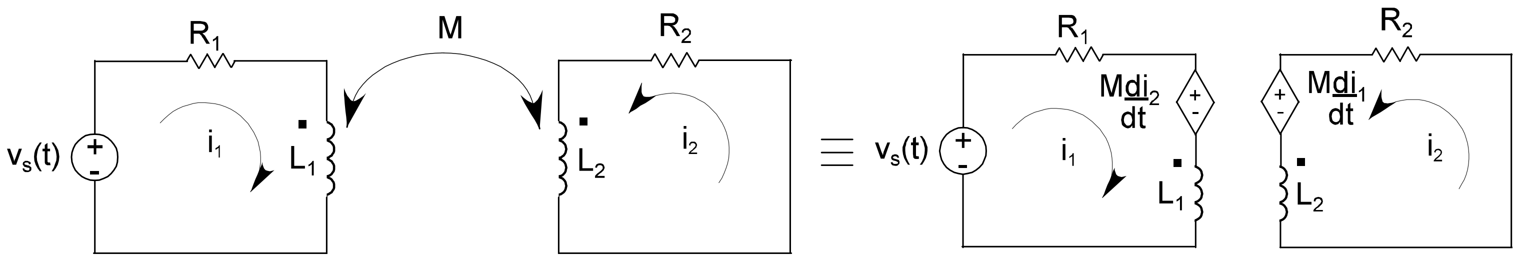
But $i_2 = 0$, so maximum stored energy is when i_1 is a maximum.

i.e. $$W_{max} = \tfrac{1}{2} \times 0.15 \times 5^2 = \boxed{1.875\text{ J .}}$$

Polarity of Induced Voltages

The polarity of the induced voltage depends on the direction of the flux that links it and the orientation of its own coil. The polarity of the induced voltage will reverse if the coil is unwound and re-wound in the opposite direction.

In circuit analysis the start of a winding is indicated by a dot. When current flows into the dot on one coil the voltage induced in the other coil is positive at that coil's dot. When current flows out of the dot on one coil the voltage induced in the other coil is negative at that coil's dot. This is shown in the circuit diagrams below.



Notice that the magnitude of the mutual voltage is always the mutual inductance times the rate of change of the current in the other coil. In the above circuit, both currents enter the coils at the dotted end; therefore, both mutual voltages have their "+" signs at the **top** because the coils have their dots at the **top**.

The mesh equations for the above circuit are:

$$\left.\begin{aligned} -v_s + i_1R_1 + L_1\frac{di_1}{dt} + M\frac{di_2}{dt} &= 0 \\ i_2R_2 + L_2\frac{di_2}{dt} + M\frac{di_1}{dt} &= 0 \end{aligned}\right\}$$

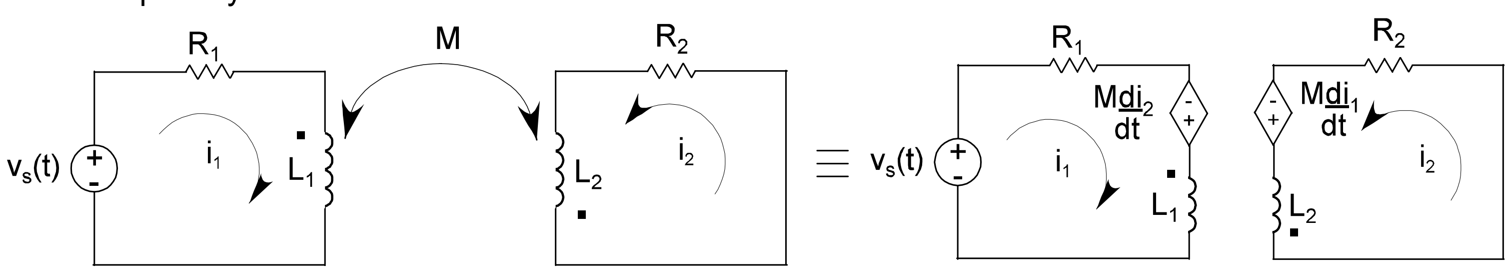
When sinusoids are applied, the phasor form of these equations should be used

The above equations in phasor form become:

$$-\boldsymbol{V_s} + \boldsymbol{I_1}R_1 + j\omega L_1\boldsymbol{I_1} + j\omega M\boldsymbol{I_2} = 0$$

$$\boldsymbol{I_2}R_2 + j\omega L_2\boldsymbol{I_2} + j\omega M\boldsymbol{I_1} = 0$$

An alternative arrangement is shown in the circuit diagrams below, notice that the polarity of coil 2 has been reversed.



Once again, the magnitude of the mutual voltage is always the mutual inductance times the rate of change of the current in the other coil. In the above circuit, i_1 enters at its dotted end; therefore the mutual voltage in mesh 2 has its "+" at the **bottom** because coil 2's dot is at the **bottom**. Since i_2 enters at its non-dotted end the mutual voltage in mesh 1 has its "+" at the **bottom** because coil 1's dot is at the **top**.

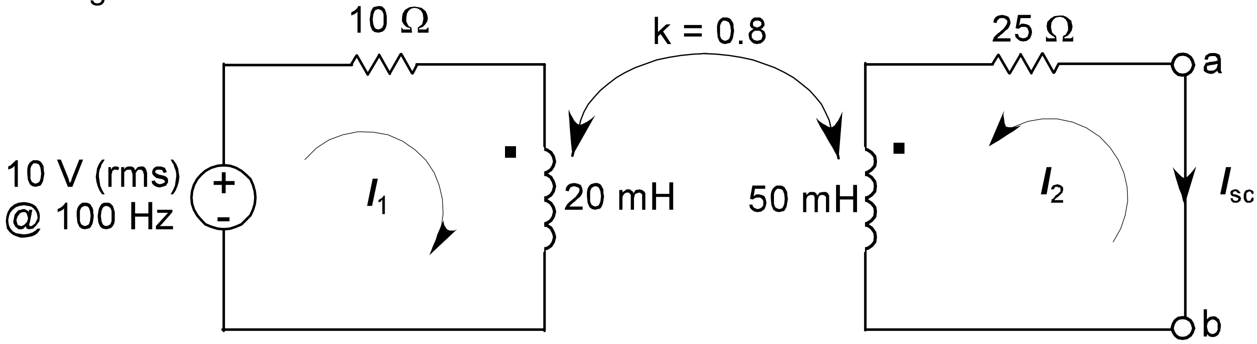
The mesh equations for the above circuit, in phasor form, are:

$$-\boldsymbol{V_s} + \boldsymbol{I_1}R_1 + j\omega L_1\boldsymbol{I_1} - j\omega M\boldsymbol{I_2} = 0$$

$$-j\omega M\boldsymbol{I_1} + \boldsymbol{I_2}R_2 + j\omega L_2\boldsymbol{I_2} = 0$$

Example

Calculate the short-circuit current I_{sc}, the source current I_1 and the Thevenin equivalent looking into terminals a-b.



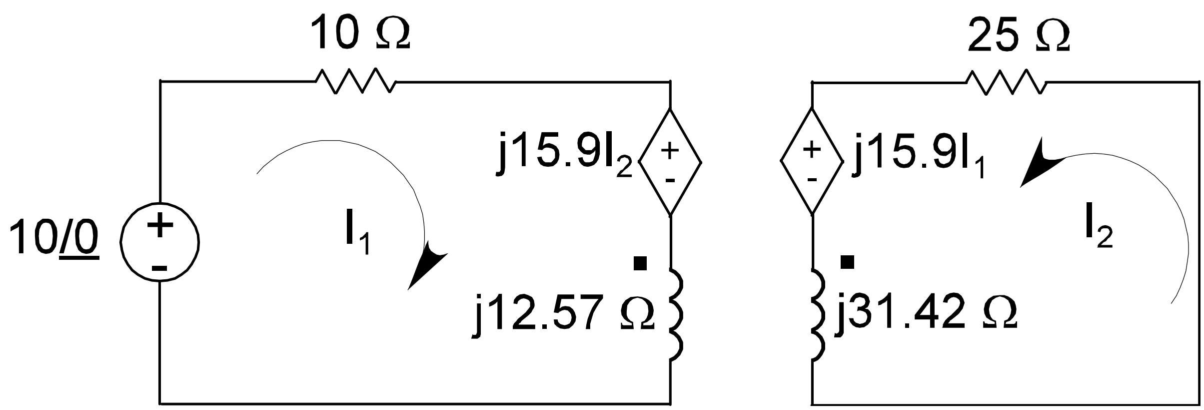
We have to start by getting the equivalent circuit, this means determining M.

$$M = k\sqrt{L_1 L_2} = 0.8\sqrt{20 \text{ x } 50} = 25.3 \text{ mH}$$

Since $\omega = 2\pi \text{ x } 100 = 628.3$ rad/s we can determine the reactances as follows:

$X_M = 15.9\ \Omega$, $\quad X_1 = 12.57\ \Omega$, $\quad X_2 = 31.42\ \Omega$

With a – b on S/C the phasor form of the circuit is:



The mesh equations are:

$$-10\angle 0 + (10 + j12.57)\boldsymbol{I_1} + j15.91\boldsymbol{I_2} = 0$$
$$j15.91\boldsymbol{I_1} + (25 + j31.42)\boldsymbol{I_2} = 0$$

These solve to give: $\boldsymbol{I_1} = 0.63\angle -28.9$ A, $\boldsymbol{I_2} = 0.25\angle -170.3$ A $= -0.25\angle 9.7$ A

And since $\boldsymbol{I_{SC}} = -\boldsymbol{I_2}$ we get: $\boxed{\boldsymbol{I_{SC}} = 0.25\angle 9.7 \text{ A}}$.

By definition: $\boldsymbol{V_T} = \boldsymbol{V_{OC}} = j15.9\boldsymbol{I_{1oc}}$, where $\boldsymbol{I_{1oc}}$ is $\boldsymbol{I_1}$ when mesh 2 is O/C ($\boldsymbol{I_2} = 0$).

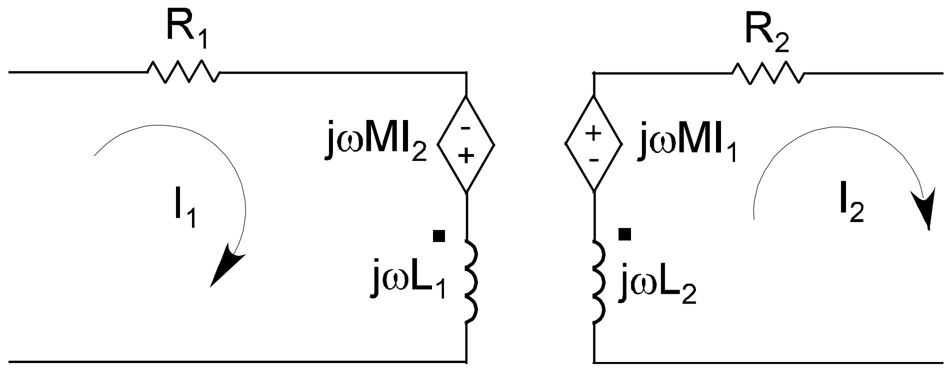
$$\boldsymbol{I_{1oc}} = \frac{10\angle 0}{10 + j12.57} = 0.623\angle -51.5 \text{ A} \qquad \therefore \qquad \boldsymbol{V_T} = j15.9 \text{ x } 0.623\angle -51.5 = \boxed{9.98.5 \text{ V}}.$$

$$Z_T = \frac{V_T}{I_{SC}} = \frac{9.9\angle 38.5}{0.25\angle 9.7} = 39.6\angle 28.8 = 34.7 + j19.1\,\Omega$$

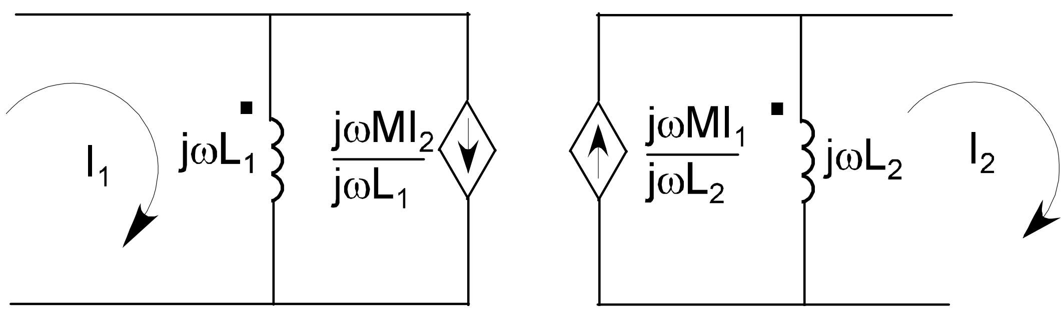
(**NOTE:** $Z_T \neq 25 + j31.42$, because of the dependent source.)

Ideal Transformer Model

We have seen the equivalent circuit for a pair of mutually-coupled coils as:



An ideal transformer has no losses, so both R's are zero, k = 1.0, and the core reluctance is zero, meaning the inductances are infinite. Applying Thevenin to Norton conversions gives:



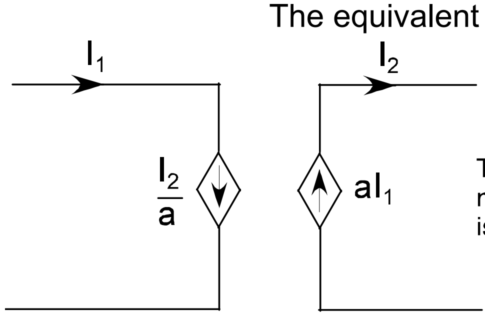
Consider the left-hand side; $j\omega L_1$ is an open-circuit because L_1 is infinite. The $j\omega$ terms in the current source cancel and we have M divided by L_1, which gives:

$$\frac{M}{L_1} = \frac{k\sqrt{L_1 L_2}}{L_1} = \sqrt{\frac{L_2}{L_1}}, \text{ because } k = 1.0$$

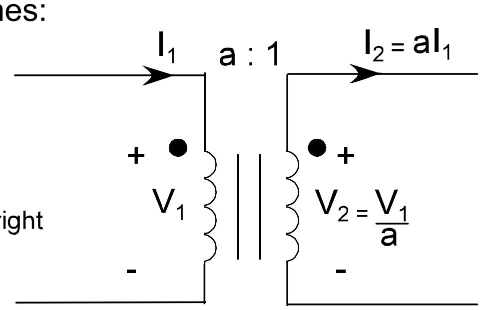
However, L_1 and L_2 are on the same core, so they see the same reluctance, $\Re$, and the outcome is:

$$\sqrt{\frac{L_2}{L_1}} = \sqrt{\frac{N_2^2/\Re}{N_1^2/\Re}} = \frac{N_2}{N_1} = \frac{1}{a}$$

The equivalent circuit becomes:



The standard notation for this is shown on the right



The two bars in between the coil symbols mean that the transformer is wound on a ferro-magnetic core. If these were not present, then the transformer would be "air-cored".

The "a : 1" symbol means that the primary coil has "a" times as many turns as the secondary coil.