Magnetostatics

Ørsted's experiment

Since the 1700's, perhaps even earlier, suspicions existed that there was some connection between electricity and magnetism. These suspicions were fed by observations of lightning.

> In 1735, a report was published discussing a curious result of lightning striking a house. The bolt traveled through a box containing a number of knives and forks. Subsequently, it was found that many of the knives were magnetized and could pick up nails. This report led Ben Franklin to conduct his well known experiment in 1751 on the magnetization of needles with the discharge of Leyden jars.*

During a course on "Electricity, Galvanism, and Magnetism,", Hans Christian Ørsted, noticed that the compass needle was deflected in the presence of an electric current. Surprised and not understanding the effect, Ørsted kept mum during the experiment and published his observations in July, 1820. Ørsted was not the first to observe and publish the effect. In 1802, Gian Domenico Romagnosi published an article in an Italian newspaper regarding the effect of an electric current upon a compass needle.**

Biot-Savart law

In September of 1820, Ørsted's observations were presented at a meeting of the French Academy, and several of the members repeated and extended his experiments.* Two of these, Jean-Baptiste Biot and Félix Savart, published what is now known as the Biot-Savart law.

* *A History of the Theories of Aether & Electricity* by Edmund Whittaker, Dover.

** http://en.wikipedia.org/wiki/Hans_Christian_%C3%98rsted

Magnetic force between two moving charges

Two moving charges, q_1 and q_2, exert a magnetic force on one another. This force is different than the Coulomb force. The formulation below is valid for $\mathbf{v}_1$ and $\mathbf{v}_2$ both << c. (light in vacuum)

$$\mathbf{F}_{m12} \propto \frac{q_1\, q_2}{R^2}\, \mathbf{v}_2 \times \left(\mathbf{v}_1 \times \mathbf{a}_R\right) \qquad (\mathbf{F}_{m12} = \text{force on } q_2,\ \mathbf{a}_R = \text{unit vector from } q_1 \text{ to } q_2)$$

in SI units, $$\mathbf{F}_{m12} = \frac{\mu}{4\pi}\,\frac{q_1 q_2}{R^2}\, \mathbf{v}_2 \times \left(\mathbf{v}_1 \times \mathbf{a}_R\right)$$

where m is the permeability (discussed further in later section) of the material between q_1 and q_2. In vacuum, $\mu = \mu_o = 4\pi\,(10^{-7})$ H/m.

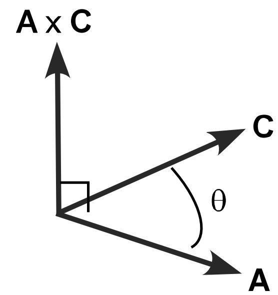
An aside: the ratio of the magnitudes of the electric and magnetic forces between charges

$$\frac{F_m}{F_e} = \frac{\mu_o\, q_1 v_1 q_2 v_2 / 4\pi R^2}{q_1 q_2 / 4\pi\,\varepsilon_o\, R^2} = \mu_o\,\varepsilon_o\, v_1 v_2 = \frac{v_1 v_2}{c^2}$$

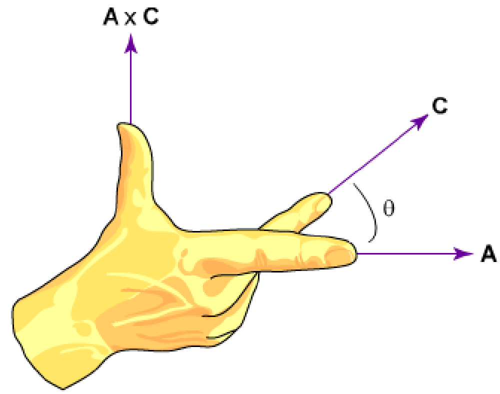
Given the minuteness of this factor, why do we ever see the magnetic force manifested between charges? The answer lies in the exquisite level of charge neutrality normal materials and in many other situations.

Example: the vector cross-product

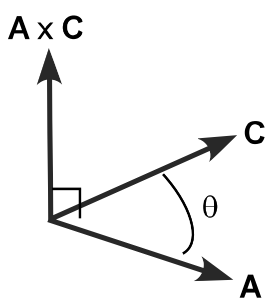
Given two vectors $\mathbf{A} = A\,\mathbf{a}_A$ and $\mathbf{C} = C\,\mathbf{a}_C$, their vector cross product $\mathbf{A} \times \mathbf{C} = AC \sin\theta\,\mathbf{a}_n$, where $\mathbf{a}_n$ is a unit vector normal to the surface defined by **A** and **C**.

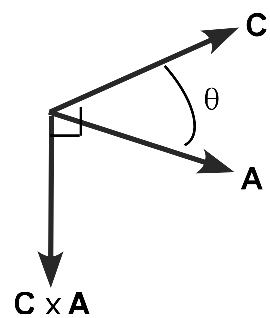


The sense of $\mathbf{a}_n$ is be determined by the right-hand rule illustrated on the right.



From this definition, it follows that $\mathbf{A} \times \mathbf{C} = -\,\mathbf{C} \times \mathbf{A}$.





The vector product can be expressed in determinate form,

$$\mathbf{A} \times \mathbf{C} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ A_x & A_y & A_z \\ C_x & C_y & C_z \end{vmatrix} = \left(A_yC_z - A_zC_y\right)\mathbf{a}_x + \left(A_zC_x - A_xC_z\right)\mathbf{a}_y + \left(A_xC_y - A_yC_x\right)\mathbf{a}_z$$

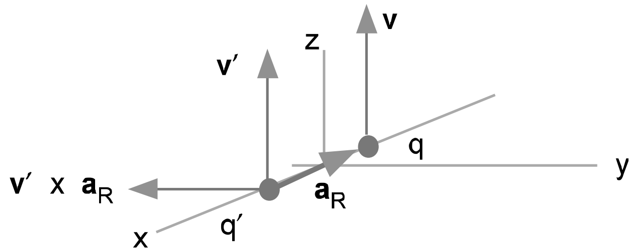
Unit vectors

(Cartesian)	$\mathbf{a}_x \times \mathbf{a}_y = \mathbf{a}_z$	$\mathbf{a}_y \times \mathbf{a}_z = \mathbf{a}_x$	$\mathbf{a}_z \times \mathbf{a}_x = \mathbf{a}_y$
(cylindrical)	$\mathbf{a}_\rho \times \mathbf{a}_\phi = \mathbf{a}_z$	$\mathbf{a}_\phi \times \mathbf{a}_z = \mathbf{a}_\rho$	$\mathbf{a}_z \times \mathbf{a}_\rho = \mathbf{a}_\phi$
(spherical)	$\mathbf{a}_r \times \mathbf{a}_\theta = \mathbf{a}_\phi$	$\mathbf{a}_\theta \times \mathbf{a}_\phi = \mathbf{a}_r$	$\mathbf{a}_\phi \times \mathbf{a}_r = \mathbf{a}_\theta$

Example: magnetic force between two moving charges

Take q′ at (x, y, z) = (x′, 0, 0) moving in the $\mathbf{a}_z$ direction and q at (-x, 0, 0) moving in the $\mathbf{a}_z$. Consider q′ and q to be positive. What is the nature of the force on q?

$$\mathbf{F}_{m\text{-}q} = \frac{\mu}{4\pi}\frac{qq'}{R^2}\,\mathbf{v} \times \left(\mathbf{v}' \times \mathbf{a}_R\right)$$



In this case, the force on q is an attractive force toward q′. With currents, the analogous observation would be that parallel currents in the same direction attract one another.

Magnetic flux density

The force on q can be rewritten in a form involving **B**, the magnetic flux density in Wb/m^2.

$$\mathbf{F}_{m\text{-}q} = q\mathbf{v} \times \left(\frac{\mu q'}{4\pi R^2}\mathbf{v}' \times \mathbf{a}_R\right) = q\mathbf{v} \times \mathbf{B}$$

This is the <u>magnetic</u> portion of the Lorentz force which acts on a charge moving in an electromagnetic field—charge q moving with velocity **v** in an electromagnetic field, **E** and **B**, experiences the a force, **F**, called the Lorentz force.

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

The magnetic field due to a differential charge $dq = \rho_v d\cancel{v}$ moving with a velocity **v** can readily be found (the differential volume is written here as $d\cancel{v}$ to avoid confusion with velocity).

$$d\mathbf{B} = \frac{\mu}{4\pi R^2}\,dq\mathbf{v} \times \mathbf{a}_R = \frac{\mu}{4\pi R^2}\,d\cancel{v}\,\rho_v\mathbf{v} \times \mathbf{a}_R$$

$$d\mathbf{B} = \frac{\mu}{4\pi R^2}\,d\cancel{v}\,\mathbf{J} \times \mathbf{a}_R = \frac{\mu}{4\pi R^2}\,I\,d\boldsymbol{\ell} \times \mathbf{a}_R$$

The permeability is a material parameter (as is conductivity and permittivity) relating the magnetic field to magnetic flux density, $\mathbf{B} = \mu\mathbf{H}$, and will be discussed more fully later.

A differential relationship for d**H** is independent of permeability.

$$d\mathbf{H} = \frac{I\,d\boldsymbol{\ell} \times \mathbf{a}_R}{4\pi R^2}$$

Biot-Savart is obtained by integrating over the complete current path.

$$\mathbf{H} = \int_{\text{current}} \frac{I\,d\boldsymbol{\ell} \times \mathbf{a}_R}{4\pi R^2}$$

Example: the magnetic fields due to other current distributions

The moving charge was expressed as a current over a path. It could also be in the form of a surface current moving on a surface or simply be in the form of a current density over some volume.

Starting from the general relationship,

$$d\mathbf{H} = \frac{dq\,\mathbf{v} \times \mathbf{a}_R}{4\pi R^2}$$

an expression can be found for the case of a surface current, $\mathbf{J}_s$, in A/m

$$d\mathbf{H} = \frac{dq\,\mathbf{v} \times \mathbf{a}_R}{4\pi R^2} = \frac{ds\,\mathbf{J}_s \times \mathbf{a}_R}{4\pi R^2}$$

$$\mathbf{H} = \iint_{\text{surface}} \frac{\mathbf{J}_s \times \mathbf{a}_R}{4\pi R^2}\,ds$$

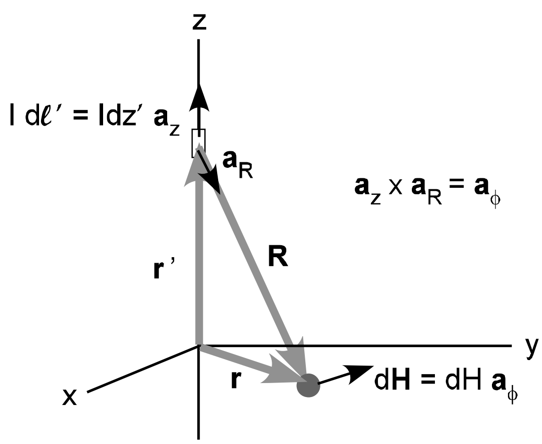
and for a general current density, $\mathbf{J}$, in A/m^2.

$$d\mathbf{H} = \frac{dq\,\mathbf{v} \times \mathbf{a}_R}{4\pi R^2} = \frac{d\cancel{v}\,\mathbf{J} \times \mathbf{a}_R}{4\pi R^2}$$

$$\mathbf{H} = \iint_{\text{surface}} \frac{\mathbf{J} \times \mathbf{a}_R}{4\pi R^2}\,d\cancel{v}$$

Example: infinite line of current ☼

Using cylindrical coordinates, we have,



$$\mathbf{r}' = z'\mathbf{a}_z \qquad \mathbf{r} = \rho\mathbf{a}_\rho + z\mathbf{a}_z$$

$$\mathbf{R} = \mathbf{r} - \mathbf{r}' = \rho\mathbf{a}_\rho + (z - z')\mathbf{a}_z$$

Using Biot-Savart for this current distribution.

$$\mathbf{H} = \int_{\text{current}} \frac{I\,d\boldsymbol{\ell} \times \mathbf{a}_R}{4\pi R^2} = \int_{\text{current}} \frac{I\,d\boldsymbol{\ell} \times \mathbf{R}}{4\pi R^3}$$

$$\mathbf{H} = \int_{z'=-\infty}^{\infty} \frac{I\,dz'\,\mathbf{a}_z \times \left[\rho\mathbf{a}_\rho + (z - z')\mathbf{a}_z\right]}{4\pi\left[\rho^2 + (z - z')^2\right]^{3/2}}$$

$$\mathbf{H} = \int_{z'=-\infty}^{\infty} \frac{I\,\rho\,dz'\,\mathbf{a}_\phi}{4\pi\left[\rho^2 + (z - z')^2\right]^{3/2}}$$

Because the range on z' is infinite, no finite value of z can affect the result.

$$\mathbf{H} = \int_{z'=-\infty}^{\infty} \frac{I\,\rho\,dz'\,\mathbf{a}_\phi}{4\pi\left[\rho^2 + z'^2\right]^{3/2}} = \frac{I\,\rho\,\mathbf{a}_\phi}{4\pi} \int_{z'=-\infty}^{\infty} \frac{dz'}{\left[\rho^2 + z'^2\right]^{3/2}}$$

From integral tables, $\displaystyle\int \frac{dz}{\left(z^2 + a^2\right)^{\frac{3}{2}}} = \frac{z}{a^2\sqrt{z^2 + a^2}}$

$$\mathbf{H} = \frac{I\,\rho\,\mathbf{a}_\phi}{4\pi}\left(\left.\frac{z}{\rho^2\sqrt{z^2 + \rho^2}}\right|_{z'=-\infty}^{\infty} \right) = \frac{I\,\rho\,\mathbf{a}_\phi}{4\pi}\left[\frac{1}{\rho^2} - \left(-\frac{1}{\rho^2}\right)\right]$$

$$\mathbf{H} = \frac{I}{2\pi\rho}\mathbf{a}_\phi$$

Ampere's law (derivation can be skimmed over without loss of continuity)
The magnetic field of a charge q moving with a velocity **v**.

$$\mathbf{H} = \frac{q\mathbf{v} \times \mathbf{a}_R}{4\pi R^2}$$

Taking the dot product of both sides with dl and integrating about a closed path.

$$\oint_{path} \mathbf{H} \bullet d\mathbf{l} = \oint_{path} \frac{q\left(\mathbf{v} \times \mathbf{a}_R\right)}{4\pi R^2} \bullet d\mathbf{l} = \frac{q}{4\pi}\oint_{path} \frac{\left(\mathbf{v} \times \mathbf{a}_R\right)}{R^2} \bullet d\mathbf{l}$$

$$\oint_{path} \mathbf{H} \bullet d\mathbf{l} = \frac{q}{4\pi}\oint_{path} \frac{\left(\frac{d\mathbf{l}'}{dt} \times \mathbf{a}_R\right)}{R^2} \bullet d\mathbf{l} = \frac{q}{4\pi}\frac{1}{dt}\oint_{path} \frac{\left(d\mathbf{l}' \times \mathbf{a}_R\right) \bullet d\mathbf{l}}{R^2}$$

For the vector triple product

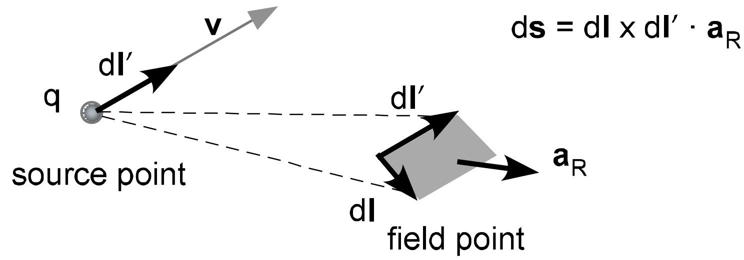
$$\left(d\mathbf{l}' \times \mathbf{a}_R\right) \bullet d\mathbf{l} = \begin{vmatrix} d\mathbf{l}_x & d\mathbf{l}_y & d\mathbf{l}_z \\ d\mathbf{l}'_x & d\mathbf{l}'_y & d\mathbf{l}'_z \\ \mathbf{a}_{Rx} & \mathbf{a}_{Ry} & \mathbf{a}_{Rz} \end{vmatrix} = -\begin{vmatrix} d\mathbf{l}'_x & d\mathbf{l}'_y & d\mathbf{l}'_z \\ d\mathbf{l}_x & d\mathbf{l}_y & d\mathbf{l}_z \\ \mathbf{a}_{Rx} & \mathbf{a}_{Ry} & \mathbf{a}_{Rz} \end{vmatrix}$$

Where the triple product is represented as a determinant which changes signs when rows are exchanged. Performing another row exchange,

$$(d\mathbf{l}' \times \mathbf{a}_R) \bullet d\mathbf{l} = \begin{vmatrix} \mathbf{a}_{Rx} & \mathbf{a}_{Ry} & \mathbf{a}_{Rz} \\ d\mathbf{l}_x & d\mathbf{l}_y & d\mathbf{l}_z \\ d\mathbf{l}'_x & d\mathbf{l}'_y & d\mathbf{l}'_z \end{vmatrix}$$

$$(d\mathbf{l}' \times \mathbf{a}_R) \bullet d\mathbf{l} = (d\mathbf{l} \times d\mathbf{l}') \bullet \mathbf{a}_R$$

Looking at this term, one can see that the triple product can be interpreted physically as outward normal projection of an elemental surface area—the familiar d**s**.

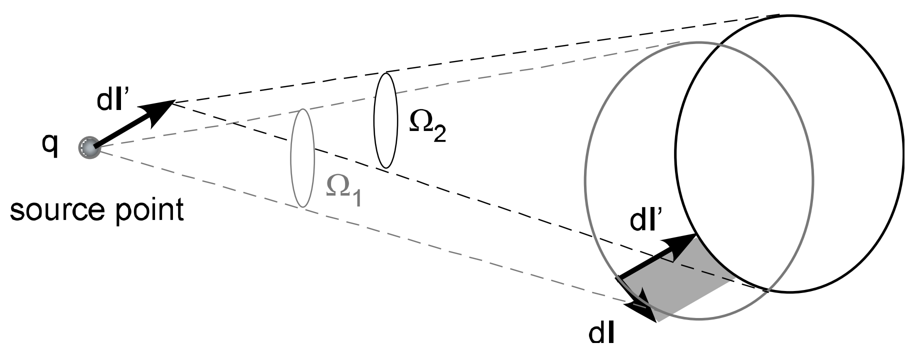


Integrating over the closed path, the surface integral is seen to be equal to the difference between the solid angle of the path at **r**' + d**l**' (Ω_2) and that at **r**' (Ω_1). Let us label this difference $d\Omega$.

$$\oint_{path} \mathbf{H} \bullet d\mathbf{l} = \frac{q}{4\pi}\frac{1}{dt}\oint_{path}\frac{d\mathbf{s}}{R^2}$$

$$\oint_{path} \mathbf{H} \bullet d\mathbf{l} = \frac{q}{4\pi}\frac{1}{dt}(\Omega_2 - \Omega_1)$$

$$\oint_{path} \mathbf{H} \bullet d\mathbf{l} = \frac{q}{4\pi}\frac{d\Omega}{dt}$$



Taking a differential of the above relationship, one arrives at the contribution of a current element.

$$d\oint_{path} \mathbf{H} \bullet d\mathbf{l} = \frac{dq}{4\pi}\frac{d\Omega}{dt} = \frac{i}{4\pi}d\Omega$$

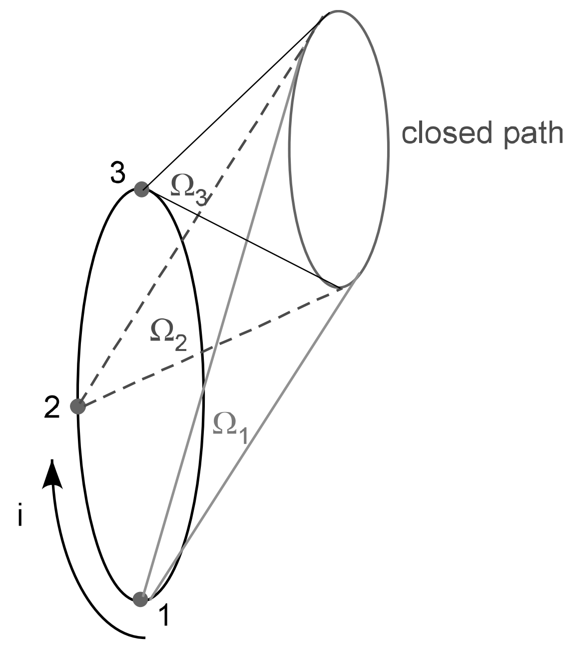
If the effect of the entire current loop is accounted for, the result is

$$\oint_{path} \mathbf{H} \bullet d\mathbf{l} = \frac{i}{4\pi}\Delta\Omega$$

There are two distinct situations—the first being the case in which the path does not encircle the current and the second in which the path does encircle the current.

Case 1: path does not enclose the current. For this case, as the current loop is traversed, the solid angle subtended by the closed path by differential current elements both increases and decreases.

If the beginning point is the same as the ending point—that is, if the path is closed path the solid angle at the beginning and at the end are equal $\Omega_1 = \Omega_2$ so that $d\Omega = 0$.

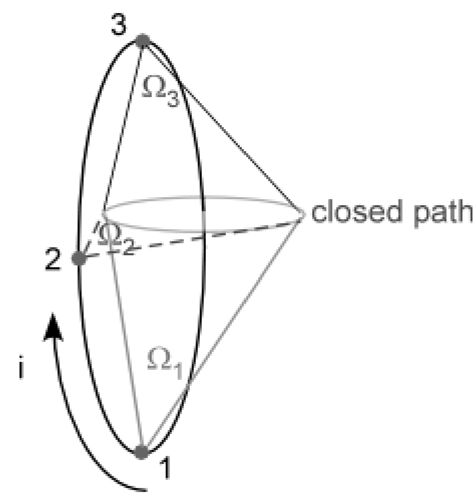


Case 2: path encloses current. In this case, much the same occurs as the current loop is traversed from points 1 to 2 to 3.

However, as the path is traversed from point 3 to point 1, the solid angle continually increases—overall, $d\Omega = 4\pi$ if the current links the path once.

If the path is linked n times then $d\Omega = n\,4\pi$.

$$\oint_{\text{path}} \mathbf{H} \bullet d\mathbf{l} = ni$$



Ampere's circuital law

The result of the above discussion is Ampere's law.

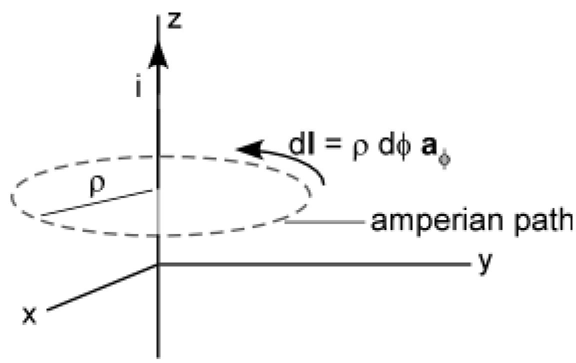
$$\oint_{path} \mathbf{H} \bullet d\mathbf{l} = i_{inclosed}$$

Ampere's law plays a role in magnetostatics similar to that Gauss' law plays in electrostatics. Just as Gauss' law provided a powerful means of finding the electric field given sufficient symmetry of the charge distribution, so Ampere's law provides a means of finding the magnetic field, again given sufficient symmetry in the source, here being a current distribution.

Example: Infinite line of current ☼

The key to using Ampere's law to determine the magnetic field is the existence of sufficient field symmetry to allow the Amperian integral to be evaluated.

In this example, combining the right-hand rule from Biot-Savart with the current's symmetry, the field must have certain symmetries.



1) $\mathbf{H}$ can only have an $\mathbf{a}_\phi$ component. $\mathbf{H} = H\mathbf{a}_\phi$
2) H can only depend on ρ

This allows the selection of an Amperian path with which to evaluate the Amperian integral.

$$\oint_{path} \mathbf{H} \bullet d\mathbf{l} = i$$

The object in choosing an Amperian path is to simplify the evaluation of the integral. Here, a circle centered on the z-axis and parallel with the z = 0 plane has $d\mathbf{l} = \rho\, d\phi\, \mathbf{a}_\phi$. The result is,

$$\oint_{path} \mathbf{H} \bullet d\mathbf{l} = \int_{\phi=0}^{2\pi} H\mathbf{a}_\phi \bullet \rho\, d\phi\, \mathbf{a}_\phi = \int_{\phi=0}^{2\pi} H\, \rho\, d\phi = 2\pi H\, \rho = i$$

The magnitude of the magnetic field is determined from an evaluation of the integral; its direction is already known to be along $\mathbf{a}_\phi$.

$$H = \frac{i}{2\pi\rho} \qquad \rightarrow \qquad \mathbf{H} = H\mathbf{a}_\phi = \frac{i}{2\pi\rho}\mathbf{a}_\phi$$

If one compares using Ampere's law to using Biot-Savart, it may be appreciated that Ampere's law is a powerful tool when there is sufficient symmetry. If such symmetry does not exist, one must then revert back to the more general Biot-Savart law.

Example: Line of current density ☼

$$\mathbf{J} = \begin{cases} 0 & \rho < a \\ 2\rho^{-0.5}\ \mathbf{a}_z\ \text{A/m}^2 & a < \rho < b \\ 0 & \rho > b \end{cases}$$

Magnetic flux

The total magnetic flux through any surface can be expressed as a surface integral of the flux density.

$$\phi = \iint_{\text{surface}} \mathbf{B} \bullet d\mathbf{s}$$

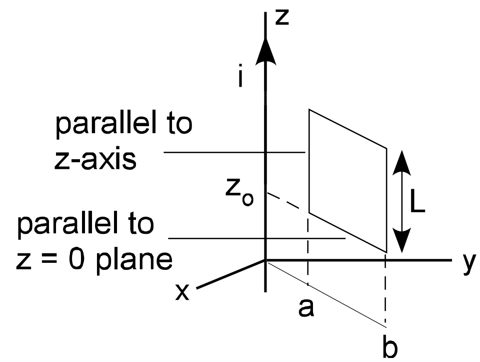
Where ϕ is the magnetic flux in Webers, and **B** is the magnetic flux density in Webers per square meter (or Tesla).

Example: flux linking an area ☼

Given current i in the $\mathbf{a}_z$ direction, find the flux linking the area shown.

Using Ampere's law, the magnetic field is determined,

$$\mathbf{H} = \frac{i}{2\pi\rho}\mathbf{a}_\phi$$



The magnetic flux density is therefore

$$\mathbf{B} = \frac{\mu i}{2\pi\rho}\mathbf{a}_\phi \quad \text{(SI units for } \mathbf{B} \text{ are Weber/m}^2\text{)}$$

A surface integral is used to find the total magnetic flux linking the area shown,

$$\phi = \iint_{\text{surface}} \mathbf{B} \bullet d\mathbf{s}$$

Notice for the area shown that there are two possible directions for the normal direction, $-\mathbf{a}_\phi$ and $\mathbf{a}_\phi$. Assuming the flux desired is that in the $\mathbf{a}_\phi$ direction, the differential vector area, d**s**, is

$$d\mathbf{s} = d\rho\, dz\, \mathbf{a}_\phi$$

The total magnetic flux is therefore

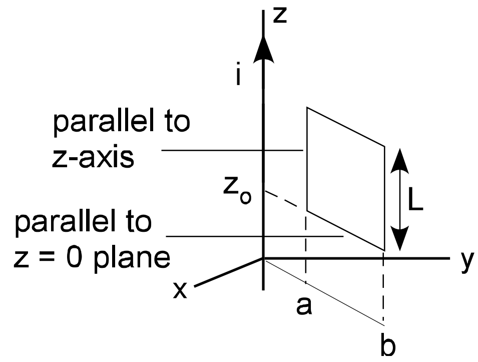
$$\phi = \int_{z=z_o}^{z_o+L} \int_{\rho=a}^{b} \frac{\mu i}{2\pi\rho}\mathbf{a}_\phi \bullet d\rho\, dz\, \mathbf{a}_\phi$$

$$\phi = \frac{\mu i}{2\pi} \int_{z=z_o}^{z_o+L} \int_{\rho=a}^{b} \frac{d\rho}{\rho} dz = \frac{\mu i}{2\pi} L \ln\left(\frac{b}{a}\right)$$

Example: flux linking an area – numeric approximation

Given the same current as above, give a numeric approximation to the total flux linking the area shown.

Divide each interval into 10 equal lengths.



Working from the integral,

$$\phi = \int_{z=z_o}^{z_o+L} \int_{\rho=a}^{b} \frac{\mu i}{2\pi\rho}\mathbf{a}_\phi \bullet d\rho\, dz\, \mathbf{a}_\phi$$

one obtains,

$$\phi \cong \frac{\mu i}{2\pi}\sum_{j=1}^{10}\sum_{i=1}^{10}\frac{\Delta\rho}{\rho_i}\Delta z = \frac{\mu i}{2\pi}\sum_{j=1}^{10}\left(\frac{L}{10}\right)\sum_{i=1}^{10}\frac{(b-a)/10}{a + i(b-a)/10 - (b-a)/20}$$

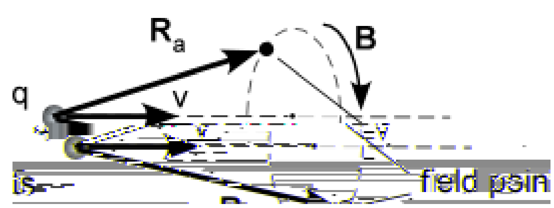
Magnetic flux lines ☼

From Biot-Savart, the fundamental relation between moving charge and the resulting field,

$$\mathbf{B} = \frac{\mu}{4\pi R^2}\, q\mathbf{v} \times \mathbf{a}_R$$

it can be seen that the lines of magnetic flux about a charge q moving with a velocity **v** are perpendicular to both the velocity and to the vector between the source and field points.

From Biot-Savart, **B** (magnetic flux density in Wb/m^2) is perpendicular to both **v** and **R** since the vector cross product of **v** and **R** is perpendicular to the plane in which **v** and **R** lie.



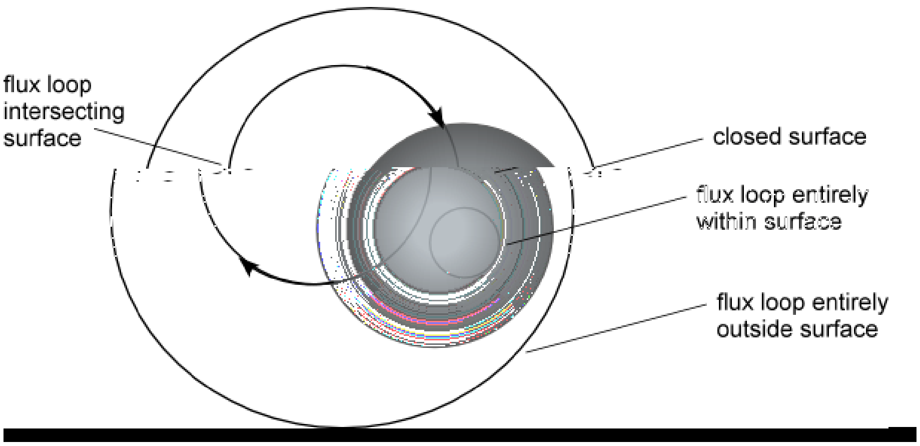
The implication is that magnetic flux lines form loops about moving charges. Magnetic flux lines have no beginning or end; they circle about moving charges. In contrast, electric flux lines begin at positive charges and end at negative ones.

The form of magnetic flux lines is due to the absence of magnetic charge. If magnetic charge existed, one would then see magnetic flux lines begin and end. Researchers have looked for these “magnetic monopoles” for many years but had not found them to until August, 2009 when the first observation of magnetic monopoles was reported in *Science* (see page 42).

Conservation of magnetic flux ☼

Since magnetic flux lines have no beginning or end, the net magnetic flux from a closed surface must be zero.

Looking at the diagram below, there are only three types of interactions between closed surfaces and closed circular loops: 1) the loop is entirely within the surface, 2) the loop is entirely outside the surface, and 3) the loop intersects the surface. There is obviously no net magnetic flux associated with interaction types 1) and 2). With 3), there is no net flux since each time the flux line enters the surface, it must also leave—thus no net flux.



Integral and point expressions ☼

The net magnetic flux leaving a closed surface must be zero. As discussed above, this is a consequence of the fact that magnetic lines are in the form of closed loops about moving charges or currents.

$$\phi_{out} = \oiint_{surface} \mathbf{B} \cdot d\mathbf{s} = 0$$

Using the divergence theorem,

$$\oiint_{surface} \mathbf{B} \cdot d\mathbf{s} = \iiint_{volume} \nabla \cdot \mathbf{B}\, dv = 0$$

This relation holds for **any** volume. That is, for any arbitrary volume, the integral of the divergence of **B** over the volume must equal zero. The only possibility for this to hold is for the divergence of **B** itself to be equal to zero.

$$\nabla \cdot \mathbf{B} = 0$$

This is the point form of the conservation of magnetic flux (sometimes referred to as Gauss' law for magnetic fields). It is the second of Maxwell's four equations discussed so far. (The first was the point form of Gauss' law, $\nabla \cdot \mathbf{D} = \rho_v$)

What is the meaning of $\nabla \bullet \mathbf{B}$? Since B is magnetic flux density, $\nabla \bullet \mathbf{B}$ must be the net magnetic flux out per unit volume.
To review a bit, recall the discussion of $\nabla \bullet \mathbf{D}$, the divergence of the electric flux density. Here, $\nabla \bullet \mathbf{D}$ is the net electric flux out per unit volume.

For the case of $\nabla \bullet \mathbf{D}$

$$\nabla \bullet \mathbf{D} = \frac{d\psi}{dv} = \frac{dq}{dv} \qquad (\psi = q \text{ from Gauss' law})$$

$$\nabla \bullet \mathbf{D} = \rho_v \qquad \left(\frac{dq}{dv} = \rho_v\right)$$

For the case of $\nabla \bullet \mathbf{B}$, since the net flux out of any volume must be zero due to the nature of magnetic flux lines.

$$\nabla \bullet \mathbf{B} = \frac{d\phi}{dv} = \frac{0}{dv}$$

$$\nabla \bullet \mathbf{B} = 0$$

Point form of Ampere's law ☼

The integral form of Ampere's law for magnetostatics has been obtained from Biot-Savart.

$$\oint_{path} \mathbf{H} \bullet d\mathbf{l} = i$$

The point form of the law can be found using Stoke's theorem and expressing the current as a surface integral of the current density. Making these substitutions,

$$\iint_{surface} (\nabla \times \mathbf{H}) \bullet d\mathbf{s} = \iint_{surface} \mathbf{J} \bullet d\mathbf{s}$$

This expression holds for *any* surface. For this to hold for any arbitrary surface is for the integrands themselves to be equal.

The result is an expression relating the curl of the magnetic field vector, **H**, to the current density, **J**.

$$\nabla \times \mathbf{H} = \mathbf{J}$$

This is the point form of Ampere's law for magnetostatics.

Magnetic forces ☼

The fundamental relation giving the force on a charge moving in an electromagnetic field is the Lorentz force law.

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

F is force on charge q
v is velocity of charge q
E is electric field
B is magnetic flux density

In many situations (motors, transformers, generators), the magnetic component is dominant. This is the case considered here.

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B}$$

Many times, especially with circuits, it is more convenient to speak of forces on currents rather than moving charges. One can readily change variables.

For an infinitesimal charge dq,

$$d\mathbf{F} = dq\ \mathbf{v} \times \mathbf{B}$$

It is often useful to express this relation in terms of currents.

$$dq\ \mathbf{v} = \rho_v\, dv\, \mathbf{v} = \rho_v\, A\, dL\, \mathbf{v} = I\, d\mathbf{L}$$

$$d\mathbf{F} = I\, d\mathbf{L} \times \mathbf{B}$$

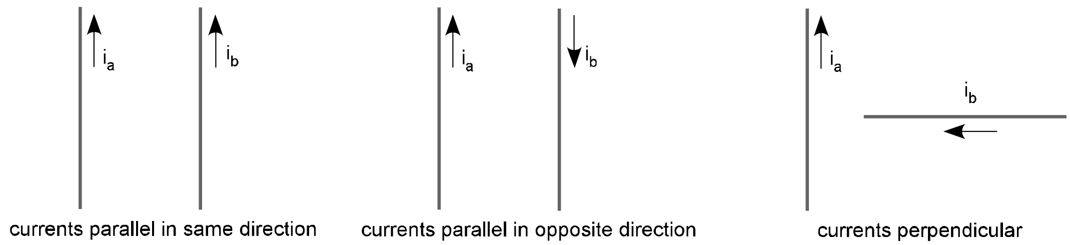
v ~ particle velocity
dv ~ differential volume
A ~ area
dL ~ differential length
d**L** ~ vector differential length (in direction of current density)
I ~ current
B ~ magnetic flux density (Wb/m^2)

Since magnetic fields are produced by currents, this force can appear in different guises.

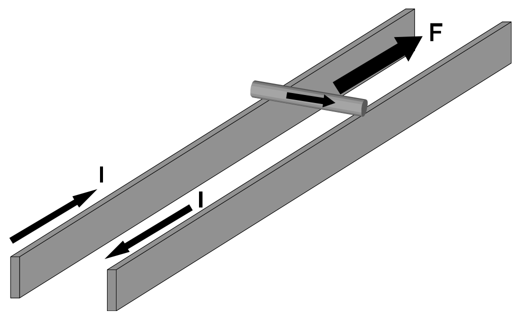
1. Between currents
2. Between currents and permanent magnets
3. Between permanent magnets

Examples

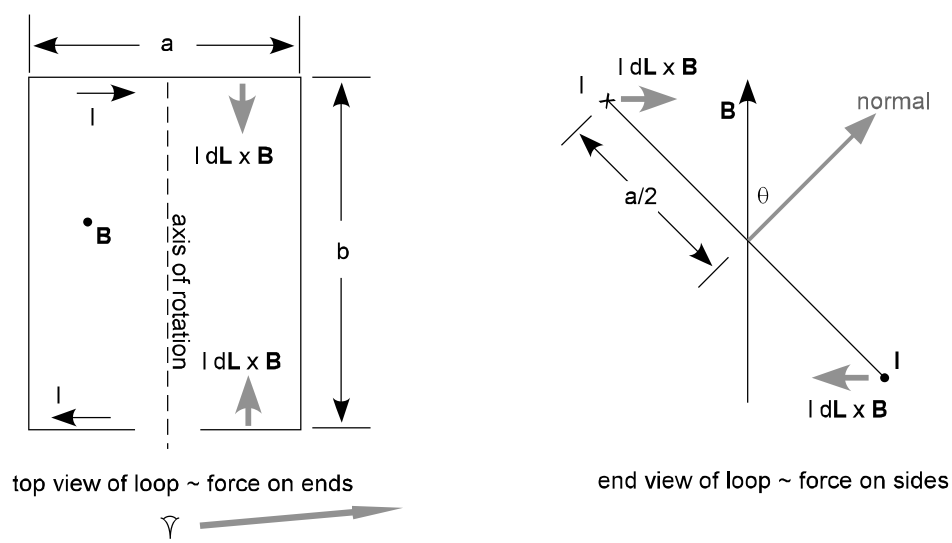
i) Do the currents below attract or repel one another?



ii) rail gun



iii) DC motor



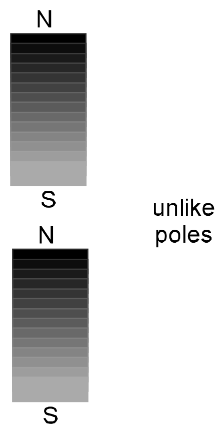
Clearly the force on the end currents exert no torque about the axis of rotation so these forces need not be considered. The magnitude of the force on each of the sides is

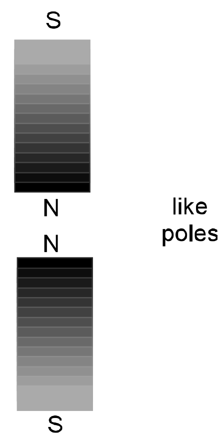
$$F = \left| \int_{side} I\, d\mathbf{L} \times \mathbf{B} \right| = I\, b\, B$$

The total torque exerted by the **two** sides about the axis of rotation is

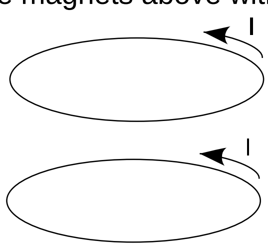
$$T = 2(IbB)\frac{a}{2}\sin\theta = I\,A\,B\sin\theta \qquad \text{where } A = ab\text{, the loop area}$$

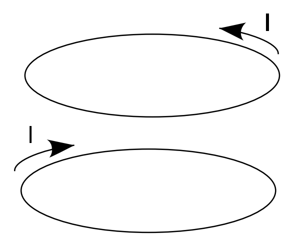
iv) Magnetics and their equivalence to current loops. Do the magnets below repel or attract one another?



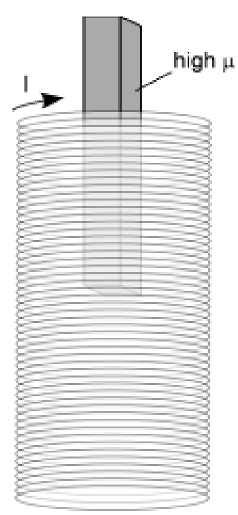


Compare the magnets above with the current loops below.





v) solenoids



Permeability ☼

The two magnetic vectors are the magnetic field, **H**, A/m, and the magnetic flux density, **B**, in Wb/m^2. The magnetic field vector is independent of permeability.

$$d\mathbf{H} = \frac{I\, d\ell \times \mathbf{a}_R}{4\pi R^2} \qquad\qquad \oint_{path} \mathbf{H} \bullet d\mathbf{l} = i_{inclosed}$$

As can be seen from these two relations, the magnetic field vector depends only on the current distribution and is independent of permeability.

The magnetic flux density vector, **B**, is related to **H** via the permeability in H/m. Permeability is the magnetic material property, and fills a role similar to that of conductivity or permittivity.

$$\mathbf{B} = \mu\mathbf{H}$$

B is a flux density. In this regard, it is similar to **D** and **J**. The unit of magnetic flux in the SI system is Weber so that the units for **B** are in Weber/m^2 or Tesla (T).

Like the magnetic field, the magnetic flux density is dependent upon the current distribution. magnetic flux density, however, also depends on permeability. Magnetic flux density is in the Lorentz force law and is, therefore, perhaps the most physical of the two vectors.

The source of the magnetic field is current. The resulting flux density may be broken into two components, the first being the flux density that would be present in vacuum and the second being the contribution of internal atomic level "currents" (electron spin or orbiting electrons) which exist or are aligned due to presence of the external field.

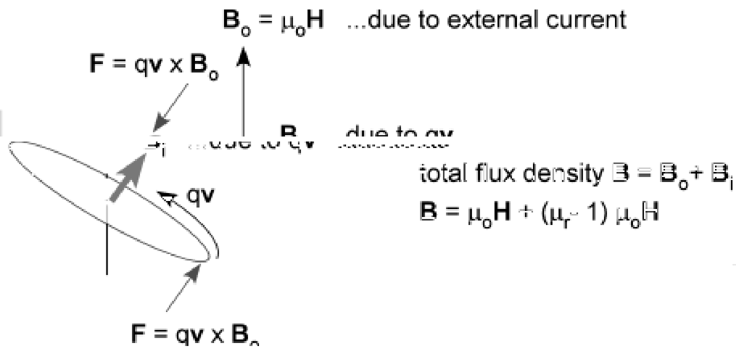
$$\mathbf{B} = \mu\mathbf{H} = \mu_r\mu_o\mathbf{H} = \mu_o\mathbf{H} + (\mu_r - 1)\mu_o\mathbf{H}$$

μ_r ~ relative permeability

μ_o ~ permeability of vacuum ($4\pi \times 10^{-7}$ H/m)

Permeability is a measure of how easily a material's internal currents are aligned in response to an applied magnetic field. These internal currents can also be referred to as internal magnetic dipoles since current loops are sources of magnetic fields with a north and a south pole – two poles, a dipole.

A material's permeability is a measure of two things – 1) the degree to which a material possesses internal magnetic dipoles, and 2) how easy these magnetic dipoles are oriented in response to the applied magnetic field.



Current loops produce magnetic fields. As already stated, another way to say this is that a current loop has a magnetic moment. Materials have three sources of magnetic moments.

1. orbiting electrons
2. electron spin
3. nuclear spin

Nuclear spin is negligible with respect to a material's gross magnetic properties and shall be neglected here.

<u>Diamagnetism</u> (universal but weak)

Without an external field the magnetic fields produced by orbital electron motion and electron spin completely cancel. Under an applied field the orbiting electrons increase or decrease their "speed" to maintain orbital stability. This unbalances the spin and orbital components slightly and produces a minor diamagnetic response.

$$\mu_r \approx 0.99999$$

All materials are diamagnetic. The effect is so weak however, that it is only important in materials in which the magnetic fields produced by orbital electron motion and electron spin completely cancel.

<u>Paramagnetism</u> (alignment of individual permanent dipoles in a material)

In paramagnetic materials, magnetic fields do not cancel (usually there are unpaired electrons). Each molecule has a net magnetic moment. With no applied field, thermal motion produces a cancellation among the dipoles. Under an external field, these dipoles align slightly to produce a magnetic field.

$$\mu_r \approx 1.00001 \text{ to } 1.004$$

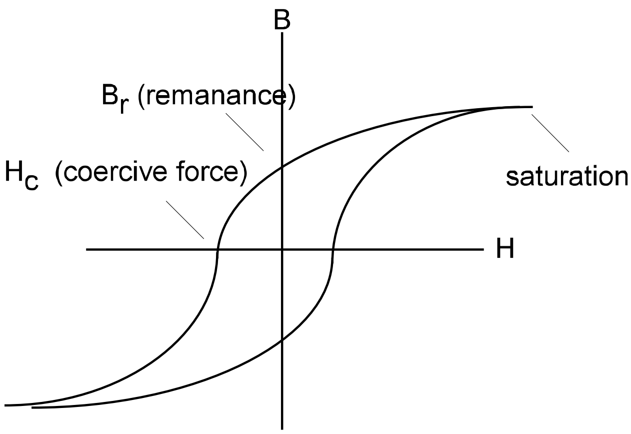
This effect, while it can be much stronger than the diamagnetic effect, is nevertheless, too weak to be of much importance in most areas of electromagnetics.

Ferromagnetism

In ferromagnetic materials, macroscopic domains (around 1 μm in linear dimension) exist having a net magnetic moment. Without an applied field, these domains are randomly oriented. Under an external field, the domains partially orient with significant effect.

μ_r can be up to 10^6 (more typically a few thousand for non-exotic materials)

Many ferromagnetic ones materials display hysteresis or magnetization or B-H curves.



NOTE
saturation is when dB/dH = 0
for B-H curve taken to *saturation*,
B_r is ref

Notice some features on this path:

1) Between point 2 and 3, H becomes zero – at this point the external field is zero. At this point, although the external field is zero, the flux density from the material is not zero. The material has become a permanent magnet. The strength of the magnet is indicated by the remanance (so named because it is the remaining flux density of a magnetized material after the external force has been removed).

   large B_r ~ strong magnet

2) At point three, the flux density of the material has become nearly zero. The material has been demagnetized, having at least once been magnetized. The value of field required to demagnetize the material is called the coercive force.

   large H_c ~ hard magnet (that is, hard to demagnetize)

Continuing on the hysteresis curve, as the applied field, H, increases from point 4, the B-H curve does not retrace the same path taken as when H was decreasing. Rather it takes the path to 5 and back to point 2.

Hysteresis is indicative of loss. In this case the loss is through lattice coupling. Lattice vibrations are induced by the magnetization/demagnetization cycles and magnetic energy is transformed into thermal energy. The magnetization-demagnetization cycle involves physical movement which couples to lattice vibrations in the material. Each time the material goes though a cycle of magnetization-demagnetization, some EM energy is transformed into thermal energy. The energy per unit volume lost for each cycle is the area of the hysteresis curve.

$$\left(\frac{\text{energy/volume}}{\text{cycle}}\right) = \oint \mathbf{H} \bullet d\mathbf{B}$$

Notice that a B-H curve with a large B_r and/or a large H_c will tend to enclose a larger area that a B-H curve with B_r and H_c both small.

The mechanism the decreasing slope of the B-H curve for larger values of H is the increasing difficulty of achieving more and more orientation of the magnetic domains. The first bit of orientation come easily – reflected in the large B-H slope (high μ), but it becomes increasingly difficult to push the dipoles into further alignment. The resistance to further alignment is largely do to Coulomb forces (aligning the dipoles for minimum magnetic energy configuration is not likely to simultaneously achieve a minimum in electric energy).

Different types of B-H curves are required for different applications. Consider those required for permanent magnets and compare these to those required by transformer cores.

Permanent magnet
Requirements would like focus on strength (higher B_r, greater strength) and hardness (higher H_c, greater hardness). The fact that these requirements would cause the material to be lossy for magnetization-demagnetization (since making B_r and H_c larger will tend to increase loop B-H hysteresis curve areas) would likely be of little concern since most permanent magnets do not undergo repeated magnetization-demagnetization. In this application, then, one can afford a lossy material.

Transformer core
What is desired for transformers core is control over permeability (often a large permeability is desired) and low loss—a thin hysteresis curve (low loss) with steep sides (high permeability).

Boundary Conditions for B ☼

The normal component of **B** is continuous across a boundary between two media. Before going on, it might be interesting to note the similarity between Gauss' law and the conservation of magnetic flux. Gauss' law states that the total electric flux coming from any closed surface is equal to the net charge within the surface.

$$\psi = q$$

In terms of the electric flux density

$$\oiint_{\text{surface}} \mathbf{D} \cdot d\mathbf{s} = q$$

The boundary condition obtained for **D** is,

$$D_{1N} - D_{2N} = \rho_s$$

Using vector notation,

$$\mathbf{a}_n \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s$$

Now compare Gauss' law to the conservation of magnetic flux. Since magnetic flux lines form closed loops, the net magnetic flux coming from a closed surface is zero.

$$\phi = 0$$

In terms of magnetic flux density,

$$\oiint_{\text{surface}} \mathbf{B} \cdot d\mathbf{s} = 0$$

Consequently, the boundary condition for B is,

$$B_{1N} - B_{2N} = 0 \qquad \rightarrow \qquad B_{1N} = B_{2N}$$

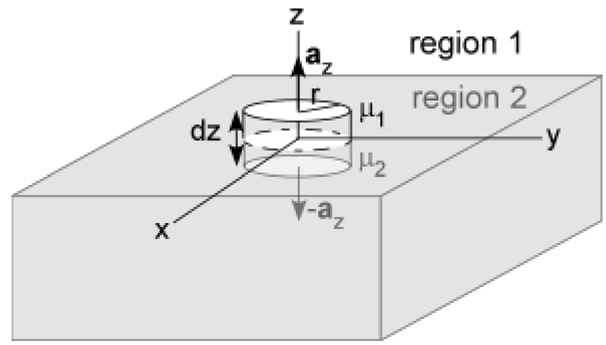
Using vector notation,

$$\mathbf{a}_n \cdot (\mathbf{B}_1 - \mathbf{B}_2) = 0$$

Derivation of boundary conditions for **B**

$$\phi_{out} = \underset{surface}{\oiint} \mathbf{B} \bullet d\mathbf{s} = 0$$

$$\iint_{top} \mathbf{B_1} \bullet d\mathbf{s} + \iint_{bottom} \mathbf{B_2} \bullet d\mathbf{s} + \iint_{sides} \mathbf{B} \bullet d\mathbf{s} = 0$$



Since the sides are infinitesimal in area, the integral over the sides is zero

$$q_{inside} = \int_{\phi=0}^{2\pi} \int_{\rho=0}^{r} \mathbf{B}_1 \bullet \rho d\rho d\phi \mathbf{a}_z + \int_{\phi=0}^{2\pi} \int_{\rho=0}^{r} \mathbf{B}_2 \bullet \rho d\rho d\phi(-\mathbf{a}_z)$$

$$B_{1z}\left(\pi r^2\right) - B_{2z}\left(\pi r^2\right) = 0$$

$$B_{1z} - B_{2z} = 0 \qquad \text{or physically, } B_{1N} - B_{2N} = 0$$

The result can be expressed compactly in vector notation using the vector normal to the interface directed from region 2 into region 1. ($\mathbf{a}_n = \mathbf{a}_z$ above)

$$\mathbf{a}_n \cdot \left(\mathbf{B}_1 - \mathbf{B}_2\right) = 0$$

Boundary Conditions for H ☼

The tangential component of **H** across two media differs by the surface current flowing at the interface. It might be interesting to review just a bit here too, and notice the similarity between Faraday's law and Ampere's law. Faraday's law states that the voltage induced (line integral of the electric field) about a closed loop is equal to the changing magnetic flux enclosed by the loop.

$$\oint_{path} \mathbf{E} \bullet d\mathbf{l} = -\frac{d\phi}{dt}$$

If the loop is infinitesimal, a finite amount of flux cannot be enclosed by the path and Faraday reads,

$$\oint_{\substack{\text{infinitesimal} \\ \text{path}}} \mathbf{E} \bullet d\mathbf{l} = 0$$

The boundary condition obtained for **E** is,

$$E_{1T} = E_{2T} \text{ or, with vector notation, } \mathbf{a}_n \times \left(\mathbf{E}_1 - \mathbf{E}_2\right) = 0$$

Compare this to using Ampere's law which states that the line integral of H about a closed path is equal to the current enclosed by the path. Here the path is an infinitesimal one on the boundary between two media. The result will be

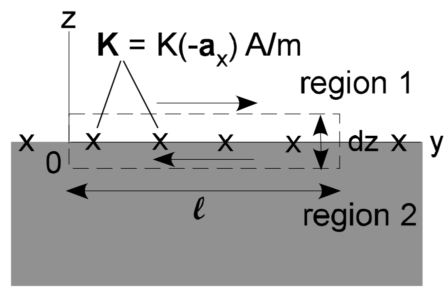
$$H_{1T} - H_{2T} = K \qquad \mathbf{a}_n \times \left(\mathbf{H}_1 - \mathbf{H}_2\right) = \mathbf{K}$$ ...**K** is surface current in A/m ($\mathbf{J}_s$ is also used)

Derivation of boundary conditions for **H**

Ampere's law states,

$$\oint_{path} \mathbf{H} \bullet d\mathbf{l} = I$$

$$\oint_{path} \mathbf{H} \bullet d\mathbf{l} = \int_{left\ side} \mathbf{H} \bullet d\mathbf{l} + \int_{top} \mathbf{H}_1 \bullet d\mathbf{l} + \int_{right\ side} \mathbf{H} \bullet d\mathbf{l} + \int_{bottom} \mathbf{H}_2 \bullet d\mathbf{l}$$



Since the side integrals are along infinitesimal lengths, they must be zero and can therefore be neglected.

$$\oint_{path} \mathbf{H} \bullet d\mathbf{l} = \int_{y=0}^{\ell} \mathbf{H}_1 \bullet dy\,\mathbf{a}_y + \int_{y=\ell}^{0} \mathbf{H}_2 \bullet dy\,\mathbf{a}_y = K\ell$$

$$H_{1y}\ell - H_{2y}\ell = K\ell \quad \rightarrow \quad H_{1y} = H_{2y}$$

Notice that $\mathbf{a}_y$ is tangential to the boundary. This result can be written more physically and independent of coordinate system choice by writing the result as

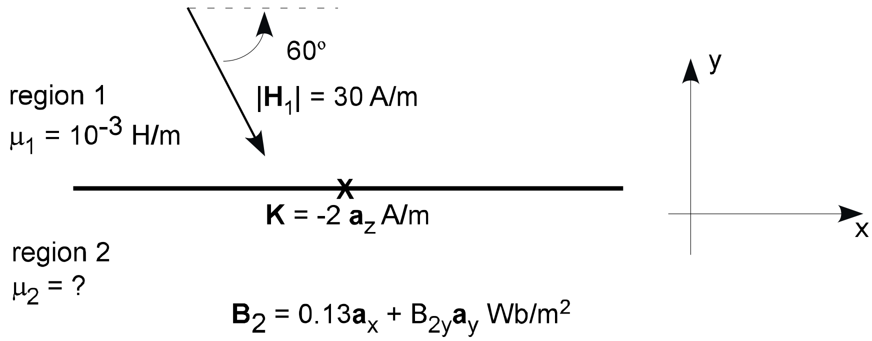
$H_{1T} - H_{2T} = K$ (positive sense of H is to the right and that of K is into the sheet)

Using vector notation, $\mathbf{a}_n \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{K}$.

Example: boundary conditions for B and H, permeability

i) find, μ_{r1} **B**$_1$ and **H**$_1$

ii) find μ_2, μ_{r2}, **B**$_2$, and **H**$_2$



i) Since **H**$_1$ and μ_1 are given, **B**$_1$ and μ_{r1} can be determined

$$\mathbf{H}_1 = 30\cos 60°\mathbf{a}_x - 30\sin 60°\mathbf{a}_y \text{ A/m}$$

$$\mathbf{H}_1 = 15\mathbf{a}_x - 15\sqrt{3}\,\mathbf{a}_y \text{ A/m}$$

$$\mathbf{B}_1 = \mu_1\mathbf{H}_1 = \mathbf{H}_1 = 0.015\mathbf{a}_x - 0.015\sqrt{3}\,\mathbf{a}_y \text{ Wb/m}^2$$

$$\mu_{r1} = \mu_1/\mu_o = 0.001/\left[4\pi\left(10^{-7}\right)\right] = \frac{10^4}{4\pi}$$

ii) Using the boundary condition $H_{1T} - H_{2T} = K$, the **a**$_x$ component of **H**$_2$ can be determined and then compared to B_{2x} to determine μ_2.

$H_{1T} - H_{2T} = K$

15 A/m $- H_{2x} = 2$ A/m (+2 A into the sheet)

$H_{2x} = 13$ A/m $= B_{2x}/\mu_2$ therefore, $\mu_2 = 0.01$ H/m

Using the boundary condition $B_{1N} = B_{2N}$, the a_y component of **B**$_2$ can be found,

$$B_{2y} = B_{1y} = -0.015\sqrt{3} \text{ Wb/m}^2$$

At this point, **B**$_2$ is known and, since μ_2 was determined in i), **H**$_2$ = **B**$_2/\mu_2$, can be found.

$$\mathbf{B}_2 = 0.13\,\mathbf{a}_x - 0.015\sqrt{3}\,\mathbf{a}_y \text{ Wb/m}$$

$$\mathbf{H}_2 = 13\,\mathbf{a}_x - \frac{0.015\sqrt{3}}{0.01}\,\mathbf{a}_y \text{ A/m} = 13\,\mathbf{a}_x - 1.5\sqrt{3}\,\mathbf{a}_y \text{ A/m}$$

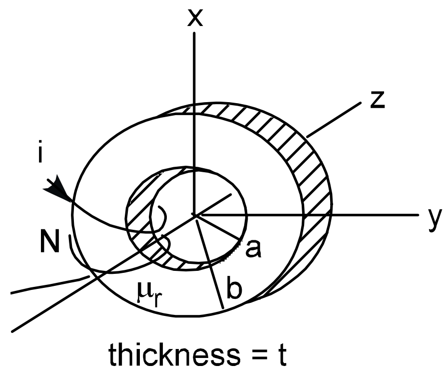
Example: toroidal core ☼

In exploring Ampere's law, it was emphasized that a symmetric current distribution gives rise to a symmetric field – e.g., an infinite line of current produces a cylindrically-symmetric field.

Another means of producing a symmetric flux density is to have a symmetrical region of high permeability. With sufficiently high permeability, the flux density will have the same symmetry (after all, for high permeability material, vast majority of the flux density is due to the alignment of the internal magnetic dipoles, not to the external current). For example, if the permeability is 1000 μ_0, then only 0.1% of the flux is due to the external current and 99.9% is due to internal circulating currents. Since the flux density is the physically meaningful quantity, it then does no harm if it is assumed the field has the same symmetry.

An important case is the toroidal core:

i) find **B** and **H** inside the toroidal core
ii) find the total magnetic flux, ϕ, in the core
iii) find **B** and **H** just inside the inner radius (in the air)
iv) find **B** and **H** just outside the outer radius (in the air)



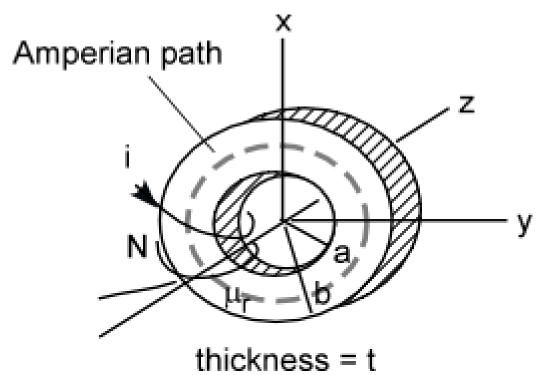
Solution:

i) From the symmetry of the core,

$\mathbf{B} \cong B\, a_\phi$, within the core, B depends only on ρ

Since $\mathbf{B} \cong B\, a_\phi$, it will be assumed that $\mathbf{H} \cong H\, a_\phi$

A suitable Amperian path is a circle centered on the z-axis for which $d\mathbf{l} = \rho\, d\phi\, \mathbf{a}_\phi$



From Ampere's law

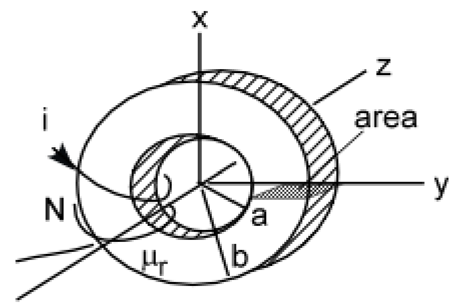
$$\oint_{path} \mathbf{H} \bullet d\mathbf{l} = i_{enclosed}$$

$$\int_{\phi=0}^{2\pi} H\, \mathbf{a}_\phi \bullet \rho\, d\phi\, \mathbf{a}_\phi = Ni$$

$$H = \frac{Ni}{2\pi\rho} \qquad \rightarrow \qquad \mathbf{H} = \frac{Ni}{2\pi\rho}\, \mathbf{a}_\phi$$

$$\mathbf{B} = \mu \mathbf{H} = \frac{\mu Ni}{2\pi\rho}\, \mathbf{a}_\phi$$

ii) The total flux can be found by integrating over the surface indicated (here it will be assumed the desired direction of flux is in the $\mathbf{a}_\phi$ direction.



$$\phi = \int_{z=z_o}^{z_o+t}\int_{\rho=a}^{b} \frac{\mu N i}{2\pi\rho}\,\mathbf{a}_\phi \bullet d\rho\, dz\, \mathbf{a}_\phi$$

$$\phi = \frac{\mu N i}{2\pi}\int_{z=z_o}^{z_o+t} dz \int_{\rho=a}^{b} \frac{d\rho}{\rho} = \frac{\mu N i}{2\pi} t \ln\left(\frac{b}{a}\right)$$

iii) At the air-core boundary on the inner radius, the field is totally tangential with zero surface current. Therefore, on the air side of the boundary, $\mathbf{H} = \frac{Ni}{2\pi a}\,\mathbf{a}_\phi$ and $\mathbf{B} = \frac{\mu_o N i}{2\pi a}\,\mathbf{a}_\phi$

iv) Likewise the field is tangential with zero surface current at the outer radius also and therefore, on the air side of the boundary, $\mathbf{H} = \frac{Ni}{2\pi b}\,\mathbf{a}_\phi$ and $\mathbf{B} = \frac{\mu_o N i}{2\pi b}\,\mathbf{a}_\phi$

Inductance ☼

Ampere's law states that there is a linear relationship between current and the magnetic field it produces.

$$\oint_{path} \mathbf{H} \bullet d\mathbf{l} = i$$

Given a linear relationship between **B** and **H** (constant permeability **B** = μ**H**), the linear relation between **H** and i can be extended to a linear relation between **B** and i.

Since flux is given through a surface integral of **B**, a linear relationship also exists between flux and current.

$$\phi = \iint_{area} \mathbf{B} \bullet d\mathbf{s}$$

Ampere's law	$\rightarrow$	linear relation between **H** and i
$\mathbf{B} = \mu\mathbf{H}$	$\rightarrow$	linear relation between **B** and i (isotropic, spatially invarient μ)
$\phi = \iint_{surface} \mathbf{B} \bullet d\mathbf{s}$	$\rightarrow$	linear relation between ϕ and i

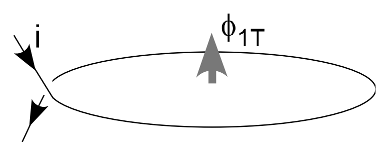
Inductance is the linear relation between, not flux, but **flux linkage**, λ, and current. Flux linkage is a new term and should be discussed before going further.

<u>Flux and flux linkage</u>

Flux linkage is the total flux enclosed by a closed path. If the path consists of multiple loops with the same flux, then the flux linkage is the product of the number of loops and the flux linked by each individual loop.

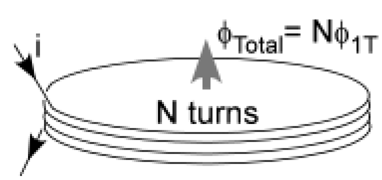
Consider the single current loop shown:

For this loop the flux from the loop due to the current i is ϕ_{1T}. The flux linkage for this one loop is $\lambda = \phi_{1T}$.



Now consider the N-turn coil consisting of N loops.

For this coil, the total flux due to the current i is $\phi_{Total} = N\phi_{1T}$. The flux linkage for the N-turn coil is $\lambda = N(\phi_{Total}) = N^2\phi_{1T}$.



The self-inductance of a coil is defined as the ratio of flux linkage to current.

$$L = \frac{\lambda}{i}$$

An aside utilizing Faraday's law (can be omitted until Faraday's law is described fully)

While the above definition for self-inductance is complete, a more satisfactory understanding can be gained by understanding of what motivates inductance. From circuits the well-known definition of inductance is

$$V = L\frac{di}{dt} \qquad \rightarrow \qquad L = \frac{V}{di/dt}.$$

That the two definitions are equivalent can be seen from Faraday's law which states that a voltage is induced in a loop inclosing a changing magnetic field.

$$V = -\frac{d\phi}{dt}$$

The negative sign is fully counted for by labeling V and the current producing ϕ (that is, i) with the passive sign convention (more on this latter when discussing Faraday's law). For now, assume PSC between V and i so that there is no negative sign.

Consider the single loop coil.

$$V_{1T} = \frac{d\phi_{1T}}{dt} = \frac{d}{dt}\left(\frac{\phi_{1T}}{i}i\right) = \frac{\phi_{1T}}{i}\frac{di}{dt}$$

Since $L_{1T} = \frac{\phi_{1T}}{i} = \frac{\lambda}{i}$ is constant and depends only geometry and permeability.

$$V = \frac{\lambda}{i}\frac{di}{dt} = L\frac{di}{dt}$$

Consider now the N-turn coil

$$V_{1T} = \frac{dN\phi_{1T}}{dt} = \frac{d}{dt}\left(\frac{N\phi_{1T}}{i}i\right) = \frac{N\phi_{1T}}{i}\frac{di}{dt}$$

$$V_{N\text{-turns}} = NV_{1T} = \frac{N^2\phi_{1T}}{i}\frac{di}{dt} = \frac{N\phi_{total}}{i}\frac{di}{dt} = \frac{\lambda}{i}\frac{di}{dt} = L\frac{di}{dt}$$

It can therefore be seen that the two definitions of self-inductance are equivalent.

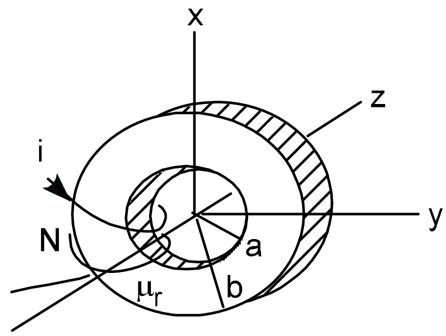
$$L = \frac{\lambda}{i} \text{ and } L = \frac{V}{di/dt}$$

Again, Faraday's law will be described in much greater detail later.

Example: self-inductance of an N-turn inductor with a toroidal core ☼

In the toroidal core inductor the total flux was found to be

$$\phi = \frac{\mu N i}{2\pi} t \ln\left(\frac{b}{a}\right)$$



The flux linkage is therefore

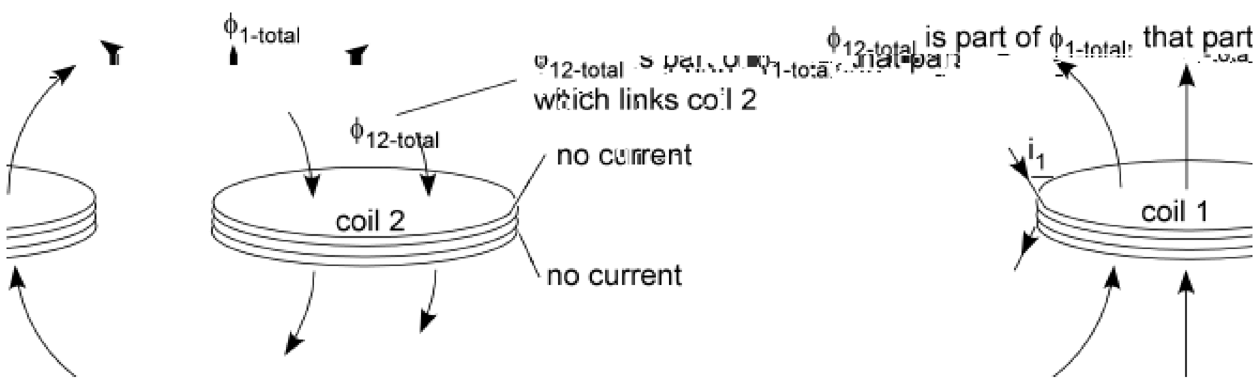
$$\lambda = \frac{\mu N^2 i}{2\pi} t \ln\left(\frac{b}{a}\right)$$

which gives a self inductance

$$L = \frac{\lambda}{i} = \frac{\mu N^2 t}{2\pi} \ln\left(\frac{b}{a}\right)$$

Mutual inductance ☼

Mutual inductance exists between coupled coils in which the flux due to a current in one coil links another coil. If part of the flux from, say coil 1, were to link a second coil, say coil 2, these two coils would be said to be mutually coupled or have mutual coupling.



The mutual coupling between the two coils is the ratio between the flux linkage of coil 1's flux to of coil 2 to the current i_1.

$$M = \frac{\lambda_{12}}{i_1} = \frac{N_2 \phi_{12\text{-total}}}{i_1}$$

It can be shown that the mutual coupling between coil 1 and coil 2 is the same as the coupling between coil 2 and coil 1. That is,

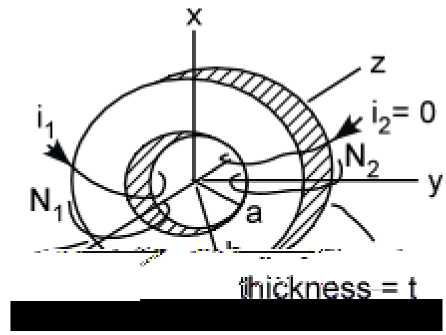
$$M = \frac{\lambda_{12}}{i_1} = \frac{N_2 \phi_{12\text{-total}}}{i_1}$$

$$M = \frac{\lambda_{21}}{i_2} = \frac{N_1 \phi_{21\text{-total}}}{i_2}$$

Example: mutual couple of two coils on a toroidal core ☼

In the toroidal core inductor with $i_2 = 0$, the flux is

$$\phi_1 = \frac{\mu N_1 i_1}{2\pi} t \ln\left(\frac{b}{a}\right)$$

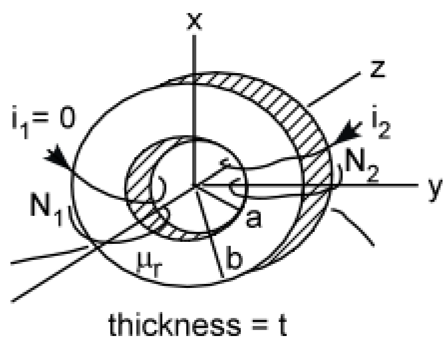


Mutual coupling between coil 1 and coil 2 is therefore,

$$M = \frac{\lambda_{12}}{i_1} = \frac{N_2 \mu N_1}{2\pi} t \ln\left(\frac{b}{a}\right)$$

On the other hand, the flux with $i_1 = 0$ and $i_2 \neq 0$ is

$$\phi_2 = \frac{\mu N_2 i_2}{2\pi} t \ln\left(\frac{b}{a}\right)$$



Mutual coupling between coil 2 and coil 1 is therefore,

$$M = \frac{\lambda_{21}}{i_2} = \frac{N_1 \mu N_2}{2\pi} t \ln\left(\frac{b}{a}\right)$$

The result in this special case—that the mutual coupling between coils 1 and 2 is the same as that between 2 and 1—is true in general.

Incremental reluctor ☼

In magnetostatics, most numerical approaches involve breaking the material into pieces that can be handled more readily.

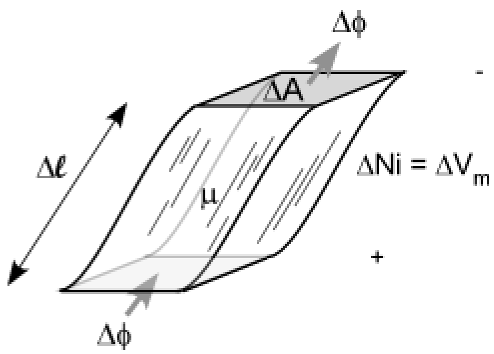
Looking at Ampere's law, one is struck by the similarity between it and a similar integral of the electric field for a potential difference.

$$\oint_{path} \mathbf{H} \bullet d\mathbf{l} = Ni \qquad \leftrightarrow \qquad \int_a^b \mathbf{E} \bullet d\mathbf{l} = V_{ab}$$

Just as $\mathbf{E} \bullet d\mathbf{l}$ can be thought of as a voltage (electromotive force) drop, so $\mathbf{H} \bullet d\mathbf{l}$ can be thought of as a drop in magnetomotive force.

The sides of the incremental reluctor are parallel to the flux density so that any flux entering the element also leaves the element. Its end surfaces are equipotential surfaces.

The element relationship for the incremental reluctor, is similar to that of the incremental resistance, the ratio of the potential difference (in this case, V_m, the magnetomotive force in Ampere) to that of flux (in this case magnetic flux).



$$\Delta\mathcal{R} = \frac{\Delta V_m}{\Delta\phi} = \frac{H\Delta\ell}{B\Delta A} = \frac{H\Delta\ell}{\mu H\Delta A}$$

$$\Delta\mathcal{R} = \frac{\Delta\ell}{\mu\Delta A}$$

Reluctance is central to magnetic circuits. The self-inductance of a coil using a magnetic core with a reluctance $\mathcal{R}$ is $L = N^2/\mathcal{R}$.

Example: toroidal core ☼

For the toroidal core, the incremental reluctor can be thin cylindrical shells of radius ρ, of height t and of thickness $d\rho$.

$$d\mathcal{R} = \frac{2\pi\rho}{\mu t d\rho}$$

These shells will be in parallel and reluctances combine as do resistors. If the reciprocal of reluctance is considered the combined reluctances will be the sum of the reciprocals of the shells.

$$d\left(\frac{1}{\mathcal{R}}\right) = \frac{\mu t d\rho}{2\pi\rho} \qquad \rightarrow \qquad \frac{1}{\mathcal{R}} = \int_{\rho=a}^{b} \frac{\mu t d\rho}{2\pi\rho} = \frac{\mu t}{2\pi}\ln\left(\frac{b}{a}\right)$$

The corresponding inductance is

$$L = \frac{N^2}{\mathcal{R}} = N^2\frac{\mu t}{2\pi}\ln\left(\frac{b}{a}\right)$$

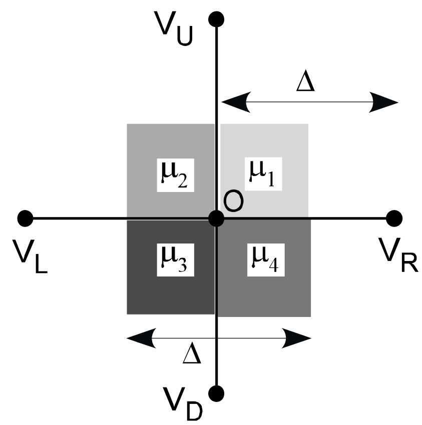
Non-homogeneous permeability

In the case of the incremental reluctor, the procedure and equations look much the same, but one must not forget that what is being found is an approximation, not to the electrostatic potential function, but to a potential corresponding to a magnetomotive potential function.

In the case of the incremental reluctor, conservation of magnetic flux gives

$$\oiint_{\text{surface}} \mathbf{B} \cdot d\mathbf{s} = 0$$

which resulted in the following equations for the node potentials,

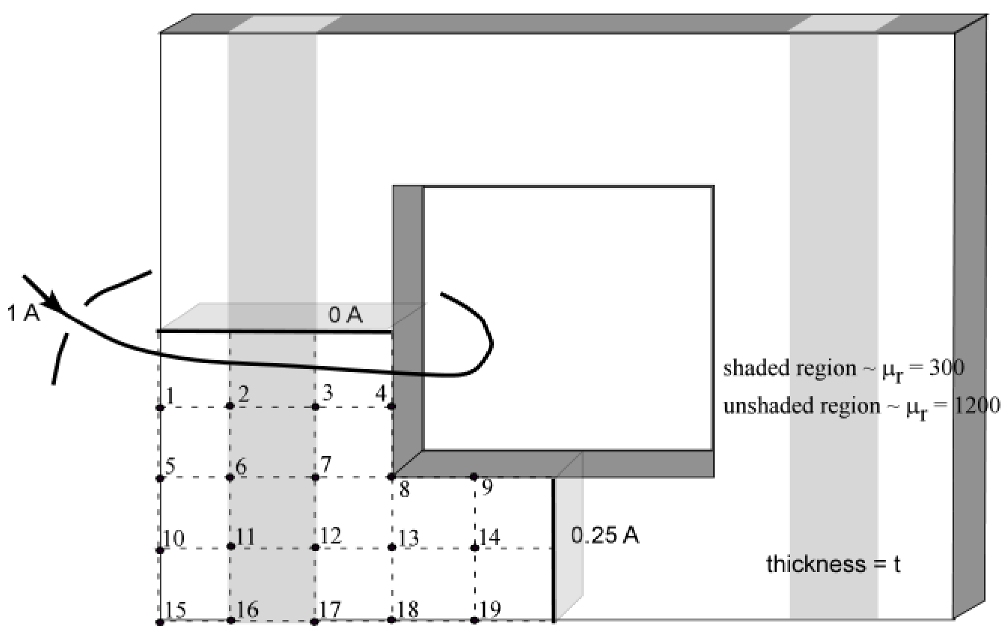


$$V_o = \frac{V_U(\mu_1 + \mu_2) + V_L(\mu_2 + \mu_3) + V_D(\mu_3 + \mu_4) + V_R(\mu_1 + \mu_4)}{2(\mu_1 + \mu_2 + \mu_3 + \mu_4)}$$

What is found is an approximation of the magnetostatic potential at discrete points from which one can approximate the magnetic fields and, knowing permeability, the magnetic flux density.

Example: two-dimensional non-homogenous core

Consider the 2D core shown, find the reluctance of the core and the inductance of the coil



Solution:
Strategy will be to find the reluctance, $\mathcal{R}_c$, and then find the inductance of an N-turn coil using

$$L = \frac{N^2}{\mathcal{R}_c}$$

Due to the core's symmetry, the reluctance of ¼ of the core can be found and the reluctance of the total core can be found by placing four quarters in series. In find the node equations, assume the permeability of the surrounding air is sufficiency low so that it may assume to be zero.

node 1: $16V_1 = 4(0\text{ A}) + 8V_2 + 4V_5$
node 2: $20V_2 = 5(0\text{ A}) + 2V_3 + 5V_6 + 8V_1$
node 3:
node 4:
node 5: $16V_5 = 4V_1 + 8V_6 + 4V_{10}$
node 6: $20V_6 = 8V_5 + 5V_2 + 2V_7 + 5V_{11}$
node 7:
node 8: $24V_8 = 4V_4 + 4V_9 + 8V_{13} + 8V_7$

and so on for all 19 nodes.

Assume all 19 node potentials have been found, the total magnetic flux can be calculated for the case of an applied magnetomotive force of 1 A-t. Using the surface between nodes 1-4 and the 0 A potential surface, one obtains

$$\phi = V_1\frac{t}{2}\ 1200\mu_o + V_2\frac{t}{2}\ (1200\mu_o + 300\mu_o) + V_3\frac{t}{2}\ (300\mu_o + 1200\mu_o) + V_4\frac{t}{2}\ 1200\mu_o$$

The reluctance of the ¼ core can then be calculated as $\mathcal{R}_{1/4} = \dfrac{0.25\text{ A-t}}{\phi}$

So that the core reluctance is

$$\mathcal{R} = \frac{1\text{ A-t}}{\phi} = \frac{1}{V_1\frac{t}{2}\ 1200\mu_o + V_2\frac{t}{2}\ (1200\mu_o + 300\mu_o) + V_3\frac{t}{2}\ (300\mu_o + 1200\mu_o) + V_4\frac{t}{2}\ 1200\mu_o}$$

For an N-turn coil about the core, the inductance would be

$$L = \frac{N^2}{\mathcal{R}} = N^2\left[V_1\frac{t}{2}\ 1200\mu_o + V_2\frac{t}{2}\ (1200\mu_o + 300\mu_o) + V_3\frac{t}{2}\ (300\mu_o + 1200\mu_o) + V_4\frac{t}{2}\ 1200\mu_o\right]$$

$$L = 300\ N^2 t\mu_o\left[2V_1 + 2.5V_2 + 2.5V_3 + 2V_4\right]$$

Magnetic energy density ☼

The energy density (W/m^3) stored in the magnetic field can be found by considering the inductance of the incremental reluctor.

$$\Delta W_m = \frac{1}{2}\Delta L\ \Delta i^2 = \frac{1}{2}\frac{N^2}{\Delta \mathcal{R}}\ \Delta i^2 = \frac{1}{2}\frac{\mu\,\Delta A}{\Delta \ell}\ \Delta (Ni)^2$$

$$\Delta W_m = \frac{1}{2}\frac{\mu\,\Delta A}{\Delta \ell}\ (H\Delta \ell)^2 = \frac{1}{2}\mu H^2\,\Delta A\,\Delta \ell$$

$$w_m = \frac{\Delta W_m}{\Delta v} = \frac{\Delta W_m}{\Delta A\,\Delta \ell} = \frac{1}{2}\mu H^2$$

$$w_m = \frac{1}{2}\mu H^2 = \frac{B^2}{2\mu} = \frac{\mathbf{B}\bullet\mathbf{H}}{2}$$

Energy density can be used to determine inductance.

$$W_m = \frac{1}{2}L i^2 \qquad \rightarrow \qquad L = \frac{2W_m}{i^2} = \frac{2\iiint_{volume} w_m\,dv}{i^2}$$

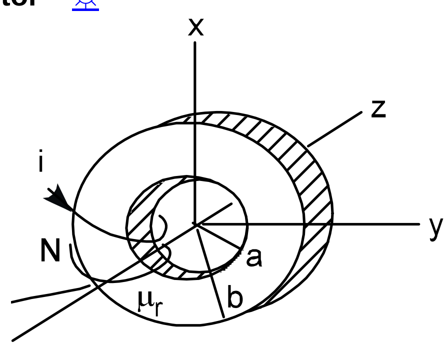
$$L = \frac{2\iiint_{volume}\frac{\mathbf{B}\bullet\mathbf{H}}{2}\,dv}{i^2} = \frac{\iiint_{volume}\mathbf{B}\bullet\mathbf{H}\,dv}{i^2}$$

Example: inductance of a toroidal-core inductor ☼

The magnetic field in the inductor core has been found to be

$$\mathbf{H} = \frac{Ni}{2\pi\rho}\ \mathbf{a}_\phi$$

Therefore, calculating inductance using the magnetic energy stored



$$L = \frac{\iiint_{volume}\mathbf{B}\bullet\mathbf{H}\,dv}{i^2} = \frac{\int_{\rho=a}^{b}\int_{z=z_0}^{z_0+t}\int_{\phi=0}^{2\pi}\left(\mu\ \frac{Ni}{2\pi\rho}\ \mathbf{a}_\phi \bullet \frac{Ni}{2\pi\rho}\ \mathbf{a}_\phi\right)\rho\ d\phi\ dz\ d\rho}{i^2}$$

$$L = \mu\left(\frac{N}{2\pi}\right)^2\int_{\rho=a}^{b}\int_{z=z_0}^{z_0+t}\int_{\phi=0}^{2\pi}\frac{d\phi\ dz\ d\rho}{\rho} = \mu\left(\frac{N}{2\pi}\right)^2\ 2\pi\ t\ \ln\left(\frac{b}{a}\right) = N^2\frac{\mu t}{2\pi}\ln\left(\frac{b}{a}\right)$$

Checking this result, one finds it the same as that found previously.

Magnetic circuits

The magnetic circuit approximation considers 1) the magnetic field, H, is parallel to the Amperian path and 2) the magnetic flux density is uniform over the cross sectional area. This approximation allows "lumped element" models to be developed for magnetic circuits.

Ampere's law states that, around a loop the total magnetomotive force "drops"

$$\oint_{path} \mathbf{H} \bullet d\mathbf{l}$$

is equal to the total of the magnetomotive force "rises" from mmf sources (current loops).

$$\oint_{path} \mathbf{H} \bullet d\mathbf{l} = i_{enc}$$

Ampere's law plays the role of Kirchoff's voltage law (KVL) in magnetic circuits. For lumped circuits, Ampere's law reads

$$\sum_i H_i\, \ell_i = \sum_j N_j\, i_j$$

Notice that the dot product of vectors can be replaced by the ordinary product of magnitudes since the field vector and the differential length vectors are parallel in magnetic circuits.

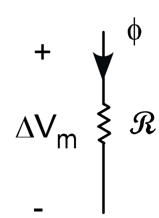
To further develop magnetic circuits, one needs to express Ampere's law using a flow variable analogous to electrical current – a quantity that obeys conservation. This will allow the use of a conservation law to play the role of Kirchoff's current law (KCL) in magnetic circuits.

Magnetic flux fills the need. The left hand side of Ampere's law, the mmf drops, are cast in terms of ϕ instead of H. The sum of these mmf drops must be equal to the mmf of the current loops.

Ampere's law therefore plays the role of KVL, conservation of magnetic flux plays the that of KCL, and reluctance fills the role played by resistance in electric circuits.

$$\sum_i H_i\, \ell_i = \sum_i \frac{B_i}{\mu_i}\, \ell_i = \sum_i \frac{\phi_i}{\mu_i A_i}\, \ell_i = \sum_i \frac{\ell_i}{\mu_i A_i}\, \phi_i = \sum_j N_j\, i_j$$

$$\sum_i \mathcal{R}_i\, \phi_i = \sum_j N_j\, i_j \qquad \text{~ magnetic "KVL", holds for all loops}$$



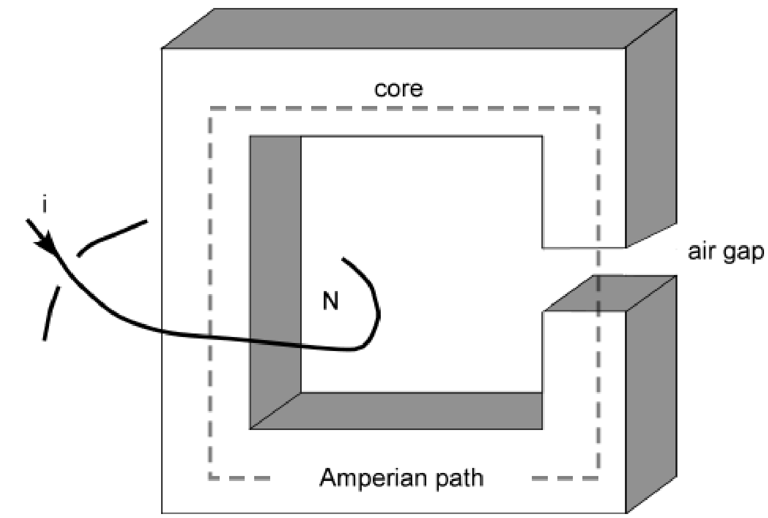
$$\mathcal{R} = \frac{\ell}{\mu A}$$

Conservation of magnetic flux at each node in the circuit yields,

$$\sum \phi_{entering} = \sum \phi_{leaving} \qquad \text{~ magnetic KCL, holds for each node}$$

Example: the magnetic circuit approximation

The assumptions behind the magnetic circuit approximation are 1) the core permeability is sufficiently high to cause the flux density follow its shape and so allows the Amperian path to be easily chosen since it is of the same shape as the core, 2) the flux density is uniform across the cross-sectional areas.



$$\oint_{path} \mathbf{H} \bullet d\mathbf{l} = \int_{core} \mathbf{H} \bullet d\mathbf{l} + \int_{gap} \mathbf{H} \bullet d\mathbf{l} = Ni$$

$$H_{core}\ell_{core} + H_{gap}\ell_{gap} = Ni$$

In terms of flux density,

$$\frac{B_{core}}{\mu_{core}}\ell_{core} + \frac{B_{gap}}{\mu_{gap}}\ell_{gap} = Ni$$

In terms of flux

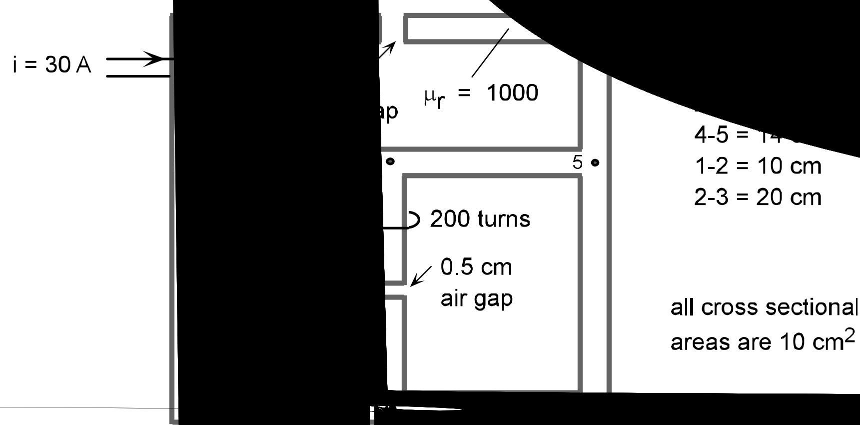
$$\frac{\ell_{core}}{\mu_{core}A_{core}}\phi_{core} + \frac{\ell_{gap}}{\mu_{gap}A_{gap}}\phi_{gap} = Ni$$

Using flux conservation ($\phi_{core} = \phi_{gap} = \phi$) and using reluctance, Ampere's law becomes

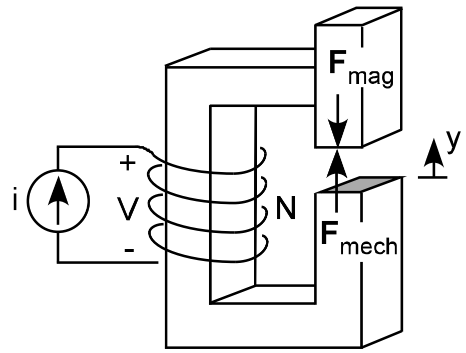
. $\left(\mathcal{R}_{core} + \mathcal{R}_{gap}\right)\phi = Ni$

Example: magnetic circuit

Analyze the circuit below us , in
circuit analysis.



Forces in magnetic circuits



Consider the force involved in magnetic actuation. Take the core and the air gap to have reluctances $\mathcal{R}_c$ and $\mathcal{R}_g$.

Using conservation of energy, consider the magnetic circuit as the system. There are two ways of increasing the energy stored in the magnetic circuit—the source and the mechanical force ($\mathbf{F}_{mech}$ = - $\mathbf{F}_{mag}$).

$$dW_{mag} = dW_{source} + dW_{mech}$$

$$\frac{i^2}{2} dL = Vi\,dt + F_{mech}dy$$

$$\frac{i^2}{2} d\left(\frac{N^2}{\mathcal{R}_g + \mathcal{R}_c}\right) = \left(N\frac{d\phi}{dt}\right) i\,dt + F_{mech}dy = Ni\,d(\psi) + F_{mech}dy$$

$$\frac{(Ni)^2}{2} d\left(\frac{1}{\mathcal{R}_g + \mathcal{R}_c}\right) = Ni\,d\left(\frac{Ni}{\mathcal{R}_g + \mathcal{R}_c}\right) + F_{mech}dy \quad \Rightarrow \quad -\frac{(Ni)^2}{2} d\left(\frac{1}{\mathcal{R}_g + \mathcal{R}_c}\right) = F_{mech}dy$$

For simplicity, assume that the core reluctance is much smaller than the gap reluctance.

$$-\frac{(Ni)^2}{2} d\left(\frac{1}{\mathcal{R}_g}\right) = -\frac{(Ni)^2}{2} d\left(\frac{\mu_o A}{y}\right) = \frac{(Ni)^2 \mu_o A}{2y^2} dy = F_{mech}dy$$

$$\mathbf{F}_{mag} = -\mathbf{F}_{mech} = -\mathbf{a}_y \frac{(Ni)^2 \mu_o A}{2y^2} = -\mathbf{a}_y A \frac{1}{2} \mu_o \left(\frac{Ni}{y}\right)^2$$

Example: electromagnetic actuators

For the common case of two air gaps, again assuming the reluctance of the core can be neglected.

$$-\frac{(Ni)^2}{2} d\left(\frac{1}{2\mathcal{R}_g}\right) = F_{mech}dy$$

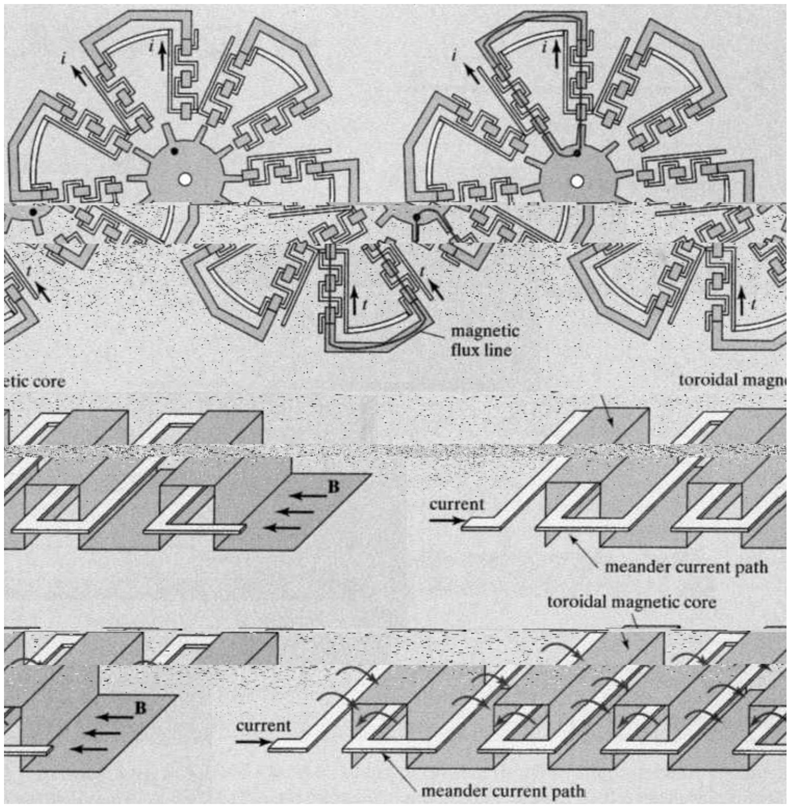
Where F_{mech} is the force on each gap.

The force of actuation from the **two** gaps is

mag - total 148d <0050>T94(a)-3775(a)]TJ EMC 1 mag - per gap mech Ni 2 o y

Magnetostatic applications examples

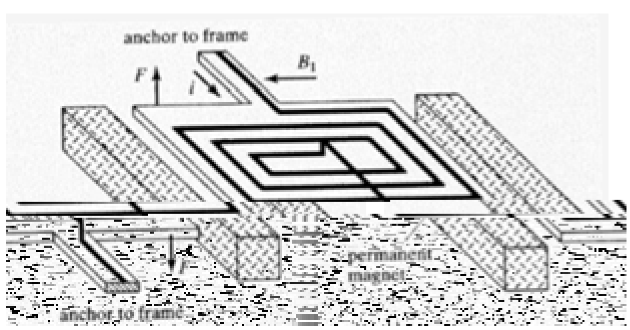
The micromotor shown below uses magnetostatics to manipulate the position of the ferromagnetic core. The operation of the motor is elegant in its simplicity. The idea is simply that forces will be in the direction to minimize the air gaps.



MICROMOTOR (from *Foundations of MEMS* by C. Liu)

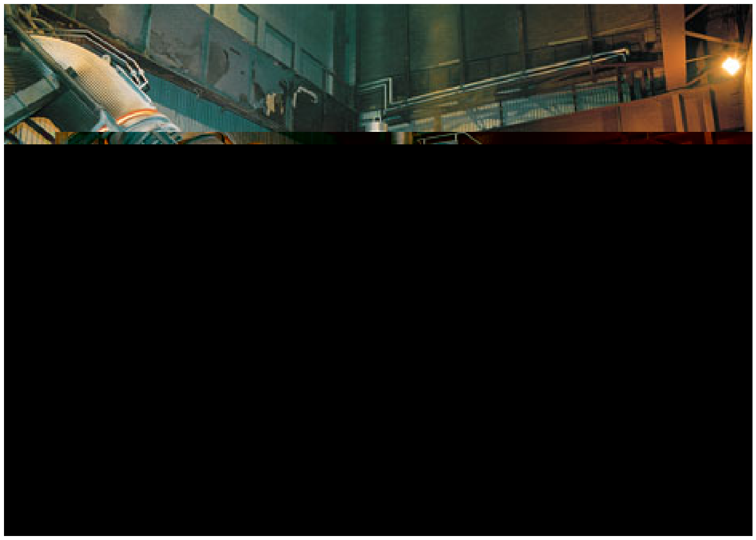
Digital light processing systems (DLPs) must efficiently be able to control light signals. One technology is from Texas Instruments and utilized electrostatically actuated micromirrors.
http://www.smalltimes.com/Articles/Article_Display.cfm?ARTICLE_ID=268087&p=109

A competing technology is to use magnetostatically actuators to control the micromirrors.



MICROMIRROR (from *Foundations of MEMS* by C. Liu)

Electric arc furnaces



http://www.steeluniversity.org/content/html/eng/default.asp?catid=25&pageid=2081271928

A few more include DC machines, electromagnets, non-destructive testing, maglev trains, all manner of solenoids, sensors.

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Dirac Strings and Magnetic Monopoles in Spin Ice $Dy_2Ti_2O_7$

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While sources of magnetic fields—magnetic monopoles—have so far proven elusive as elementary particles, several scenarios have been proposed recently in condensed matter physics of emergent quasiparticles resembling monopoles. A particularly simple proposition pertains to spin ice on the highly frustrated pyrochlore lattice. The spin ice state is argued to be well-described by networks of aligned dipoles resembling solenoidal tubes—classical, and observable, versions of a Dirac string. Where these tubes end, the resulting defect looks like a magnetic monopole. We demonstrate, by diffuse neutron scattering, the presence of such strings in the spin ice $Dy_2Ti_2O_7$. This is achieved by applying a symmetry-breaking magnetic field with which we can manipulate density and orientation of the strings. In turn, heat capacity is described by a gas of magnetic monopoles interacting via a magnetic Coulomb interaction.