Charge vs. mass

Compare the repulsion of the Coulomb force between two electrons to their gravitational attraction, let the separation be 1 m.

$$F_g = 6.673\left(10^{-11}\right)\frac{m^3}{kg\ s^2}\ \frac{\left[9.11\left(10^{-31}\right)kg\right]^2}{(1\ m)^2} = 5.54\left(10^{-71}\right)\ N$$

$$F_c = \frac{\left[1.6\left(10^{-19}\right)C\right]^2}{4\pi 8.854\left(10^{-12}\right)\ F/m\ (1\ m)^2} = 2.3\left(10^{-28}\right)\ N$$

$$\frac{F_c}{F_g} \cong 4.15\left(10^{42}\right)$$

With this enormous factor, it would be entirely possible that we'd be unaware of gravity. Obviously we are aware—why? The reason is that gravity is noticed and does matter is because most materials are, electrically, almost perfectly neutral. In fact, charge neutrality is so finely balanced at the macroscopic scale that we easily observe the effects of magnetic fields, which are actually a relativistic effect of moving charge (electromagnetics had already accounted for special relativity!)

Charged people: If two average-sized people were separated by 1 m, each having just 1% more electrons than protons, what would be repulsive force between them? Greater than Moench Hall? *Yes.* Greater than Vigo county down the depth of 1 mile? *Yes.* Greater than the United States down to a depth of 10 miles? *Yes.*

In fact, the force would be greater than the "weight" of the entire earth!

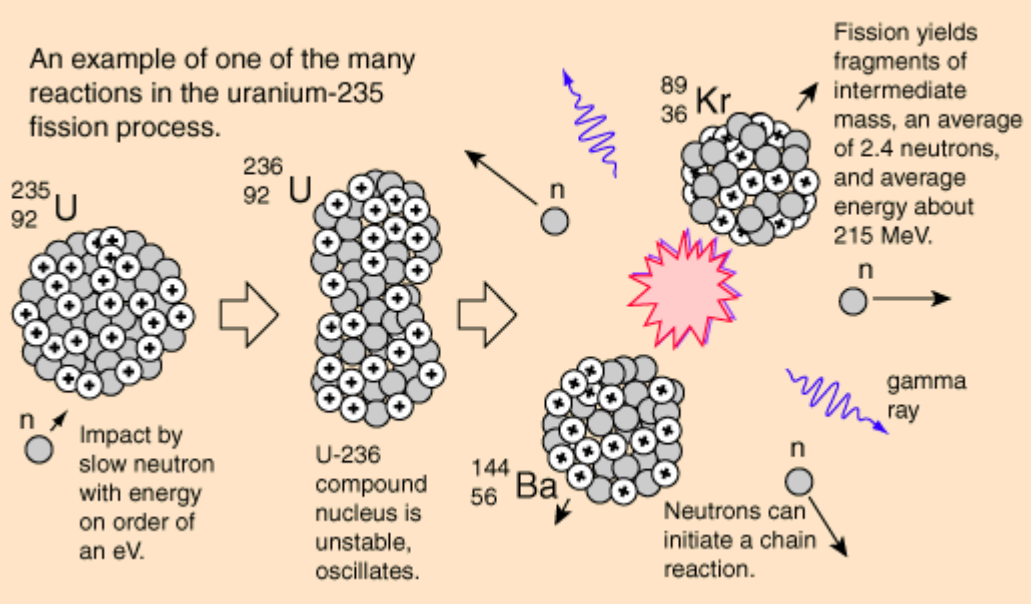
What provides atoms with their stability? The attraction between the positive nuclei and the negative electrons would, left to itself, cause the electrons to collapse upon the nuclei (in which case, our world would not be nearly as roomy) is balanced by the uncertainty principle from quantum mechanics (an electron's mean square momentum grows with increased confinement). This balance holds atoms together, but prevents them from collapsing on themselves. Moving up from individual atoms, all of chemistry is the interaction of orbital electrons.

How about the nucleus? What prevents the nucleus, made of electrically neutral neutrons and positive protons from flying apart? The answer is the balance between the repulsive Coulomb force (varying as $1/r^2$) and the attractive strong force (which is much shorter range). These two forces work together to produce stable nuclei.

The interaction of time-varying electric and magnetic fields result in electromagnetic waves, which can actually travel through **empty space**. Ask that of heat, sound, or any other type of mechanical energy!

Nuclear energy

Consider a U-235 atom undergoing fission



What is the source of the energy? Answer: largely electrostatics.

What is the potential energy of a Barium (AN = 56) atom separated by 10 fm from a Krypton (AN =36) atom?

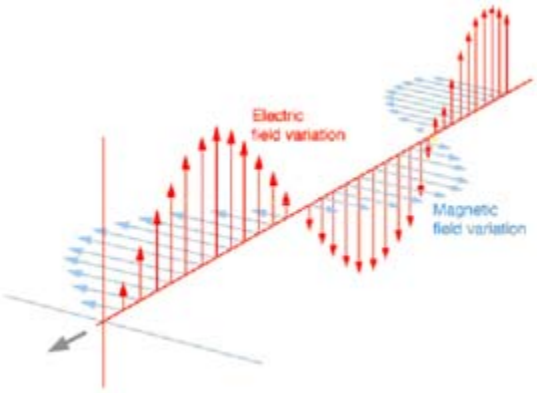
$$PE \simeq \frac{36(56)\left[1.6\left(10^{-19}\ C\right)\right]^2}{4\pi\left(\frac{10^{-9}}{36\pi}\frac{F}{m}\right)\left(10^{-14}m\right)} = \frac{(9)36(56)\left[1.6\left(10^{-19}\ C\right)\right]^2}{10^{-23}F} = (9)36(56)1.6^2\left(10^{-15}\right)\ CV$$

$$PE \simeq 4.6\left(10^{-11}\right)\ J = 2.9\left(10^{8}\right)\ eV = 290\ MeV$$

This energy is in the ball park! The diameter used, after all, is just a reasonable rough estimate for a nucleus with over 200 nucleons—the important point here is that the "nuclear energy" in fission is seen to be largely due to the Coulomb force—a "Coulomb spring" is kept coiled by the nuclear strong force until released. This happens when a neutron taps the U-235 nucleus causing an oscillation which allows the proto-barium and proto-krypton nuclei to separate beyond the ability of the very short range nuclear force to hold them together.

Electromagnetic waves

There are two keys to electromagnetic wave propagation. The first is that a time-varying magnetic field produces an electric field. The second is that a time-varying electric field produces a magnetic field. These two effects bootstrap themselves to produce a traveling electromagnetic wave.



The study of electromagnetics provides the foundation for photonics, wireless, antennas, electrical power, microwaves & RF, and high-speed circuits. The enormous topic of lumped element circuit analysis (lumped elements, KVL, and KCL) is an approximation to the electromagnetic equations, the quasi-static approximation.

The history of electromagnetics is a rich one. Following are a few random notes:

- Electron comes from the Greek for amber, Greek word for amber, ηλεκτρον. If amber is rubbed with a cloth or with fur, it aquires an electrical charge.
- An old story, probably apocryphal, goes that a Greek shepard, Magnus, noticed the iron nails in his sandles were attracted by some black stones—later called loadstones.
- Early electrical workers learned to store charge in Leyden jars in which the "electrical fluid" was thought to "condense."
- To obtain larger stores of charge, these Leyden jars were often arranged in rows, which Ben Franklin thought looks like "batteries" – batteries of canon that is.
- Volta used stacks of dissimilar metals separated by a conductiving fluid to produce the first "voltaic pile", what we'd today refer to as a battery.
- Early workers notices that, upon lightning striking a house which took a course through the cupboards, many knives and forks were melted, but others were found to be magnetized.
- In the 1860's James Clerk Maxwell presented his electromagnetic theory.
- In the 1880's Heinrich Hertz confirmed the existance of the EM waves.

The classic history of electromagnetism is a superb work by Edmund Whittaker, *A history of the theories of Aether & Electricity*, which takes the topic to 1926.

Selected online sources:

http://www.abdn.ac.uk/physics/px4006/histem.pdf
http://history.hyperjeff.net/electromagnetism
http://www.electromagnetics.biz/History.htm

Electromagnetics is fundamental to many areas of science and technology. It is a foundational topic in electrical engineering and provides a basis for advanced practice.

Knowledge of electromagnetics will maintain its utility even as technology constantly changes – indeed, its importance will likely grow. Electromagnetics plays a role in most technologies in electrical engineering and physics such as

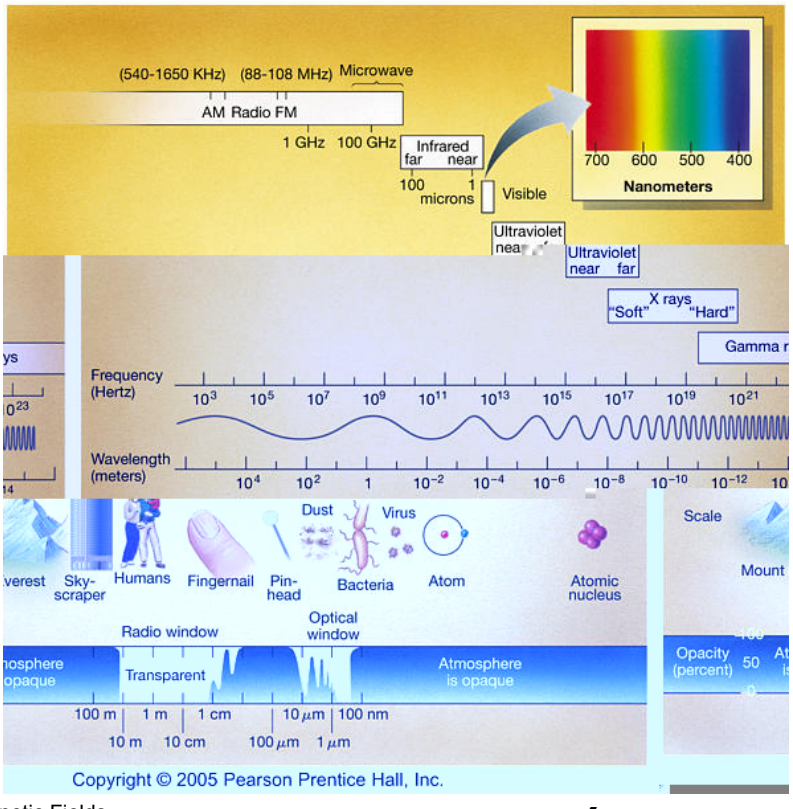
1. semiconductor devices (LEDs, diode lasers, transistors, diodes, etc.)
2. optics and optoelectronics
3. high-speed electronic systems
4. electrical machines and power
5. antennas and wireless
6. sensors (resistive, magnetic, capacitive, optical)
7. electromechanical systems, MEMS (sensors, actuators, switches)

Increasing frequencies used in high-speed design has made it critical that engineers in circuit design and layout develop an understanding of electromagnetics (EM). At higher frequencies area-fill capacitance and connection inductance can no longer be ignored, traces become transmission lines, conductors become effective antennas. At higher frequencies, simple lumped-element models become inadequate and electromagnetics is necessary to simply understand circuit behavior.

Many of these effects become evident when system dimensions are comparable to signal wavelength. Consider the table and graphic below showing wavelengths for an EM wave traveling at a speed of $3(10^8)$ m/s as the frequency is varied.

frequency vs. wavelength

3 MHz → 100 m
300 MHz → 1 m
3 GHz → 0.1 m = 10 cm
30 GHz → 1 cm



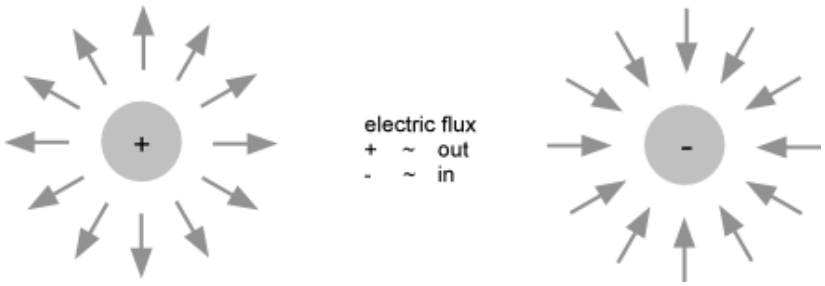
At 3 MHz, most systems are much smaller than the wavelength. At gigahertz frequencies, this condition often no longer holds, and one must resort to electromagnetic fundamentals to understand system behavior.

Our study will define relationships between the field sources (charges and currents) and the resulting electric and

magnetic fields. Fundamental quantities like energy, force, voltage and current and their relationship to the EM fields will be discussed. The relation of the EM fields to electrical parameters such as resistance, capacitance and inductance will be explored as will the consequences of wave propagation for time-varying fields.

The idea of a **electromagnetic field** began with Michael Faraday in the early 1800's who pictured flux lines emanating from electric charge. The density of the flux lines was associated with field strength, which, together with their direction constitutes a vector field. The electric and magnetic fields, vector-valued functions of position and time, are produced by charges and currents.

Michael Faraday pictured electric flux lines emanating from electric charges.



He pictured magnetic flux lines surrounding currents.

Vector notation

Vector quantities will be bold type. $\mathbf{F} = F\mathbf{a}_F$ denotes the vector **F** which consists of a magnitude F in the direction of the unit vector $\mathbf{a}_F$.

In class and videos, vectors will be underlined and unit vectors are denoted by a carrot. $\underline{F} = F\hat{a}_F$ is read that the vector $\underline{F}$ has a magnitude F and is directed along the unit vector $\hat{a}_F$.

Remember to use vector notation properly in all your work—will help both your understanding and your grade!

Some necessary mathematical fundamentals include

1. coordinate systems (rectangular, cylindrical, spherical)
2. vector operations (gradient, divergence, curl)
3. vector calculus (line integrals, surface integrals)

Rectangular (Cartesian) coordinates

x distance along x-axis or distance from y-z plane
y distance along y-axis or distance from z-x plane
z distance along z-axis or distance from x-y plane

position vector

$\mathbf{r} = x\,\mathbf{a}_x + y\,\mathbf{a}_y + z\,\mathbf{a}_z$

differential lengths

dx, dy, dz vector differential length: $d\mathbf{l} = dx\,\mathbf{a}_x + dy\,\mathbf{a}_y + dz\,\mathbf{a}_z$

differential areas

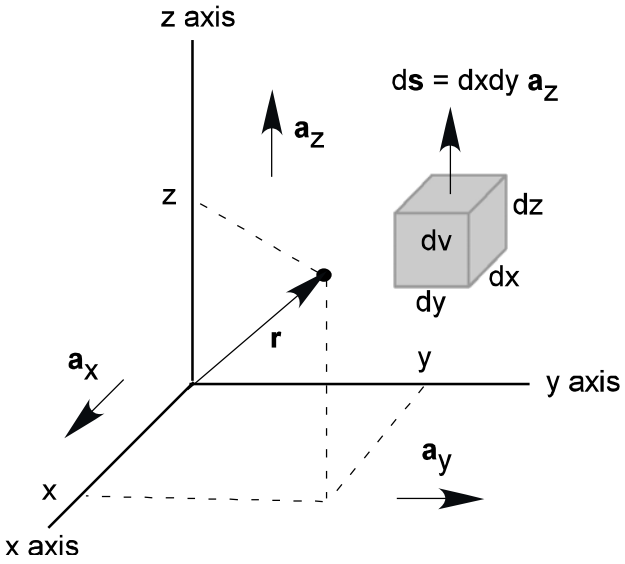
(ds) dxdy, dydz, dzdx
(d**s**) $\pm dxdy\,\mathbf{a}_z$, $\pm dydz\,\mathbf{a}_x$, $\pm dzdx\,\mathbf{a}_y$

differential volume

dxdydz

variable range

$-\infty$ to ∞ for x, $-\infty$ to ∞ for y, $-\infty$ to ∞ for z



Rectangular coordinates have some attractive properties and are often the standard coordinates used when symmetry considerations do not urge the use of another system.

1. The directions of the unit vectors are constant and not functions of position.
2. The differential elements are not functions of the coordinates.
3. The unit vectors are mutually orthogonal ($\mathbf{a}_x \cdot \mathbf{a}_y = \mathbf{a}_y \cdot \mathbf{a}_z = \mathbf{a}_z \cdot \mathbf{a}_x = 0$)

Cylindrical coordinates

- ρ distance from z-axis
- ϕ angle from +x-axis (to determine sense, align the thumb of your right hand in along the +z-axis, direction of positive rotation is along your fingers)
- z distance along z-axis or distance from x-y plane

position vector

$\mathbf{r} = \rho\,\mathbf{a}_\rho + z\,\mathbf{a}_z$

- Note the position vector does **not** explicitly involve ϕ but is present via the $\mathbf{a}_\rho$ unit vector.
- In the cylindrical coordinate system, both $\mathbf{a}_\rho$ and $\mathbf{a}_\phi$ are functions of position.

differential lengths

$d\rho$, $\rho d\phi$, dz vector differential length: $d\mathbf{l} = d\rho\,\mathbf{a}_\rho + \rho d\phi\,\mathbf{a}_\phi + dz\,\mathbf{a}_z$

differential areas

(ds) $\rho d\rho d\phi$, $d\rho dz$, $\rho d\phi dz$

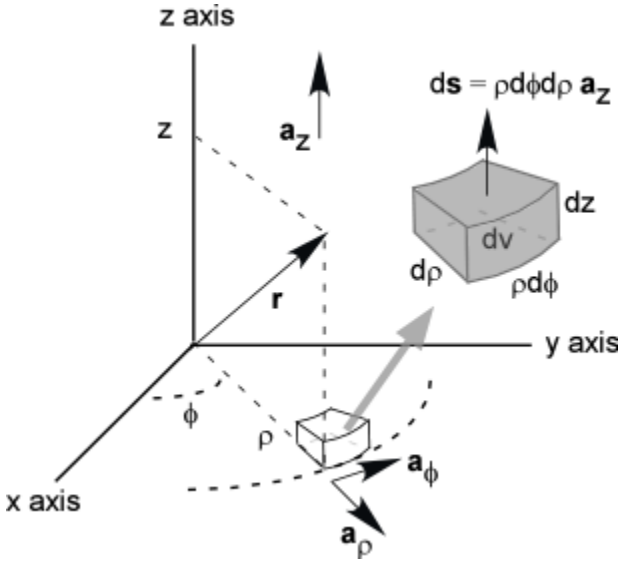
(d**s**) $\pm\rho d\rho d\phi\,\mathbf{a}_z$, $\pm d\rho dz\,\mathbf{a}_\phi$, $\pm\rho d\phi dz\,\mathbf{a}_\rho$

differential volume

$\rho d\rho d\phi dz$

variable range

0 to ∞ for ρ, 0 to 2π for ϕ, $-\infty$ to ∞ for z



Cylindrical coordinates are useful in systems with cylindrical symmetry.

1. The unit vectors in the cylindrical coordinate system are mutually orthogonal.
   $\mathbf{a}_\rho \cdot \mathbf{a}_\phi = \mathbf{a}_\phi \cdot \mathbf{a}_z = \mathbf{a}_z \cdot \mathbf{a}_\rho = 0$
2. Unlike the rectangular coordinate system, the unit vectors are not all of constant direction. The direction for $\mathbf{a}_\rho$ and $\mathbf{a}_\phi$ depend on position.

Spherical Coordinates

r distance from origin

θ angle formed starting from positive z-axis and moving to position vector

ϕ angle from the positive x-axis (to determine sense, align the thumb of your right hand in along the +z-axis, direction of positive rotation follows your fingers)

position

$\mathbf{r} = r\,\mathbf{a}_r$

differential lengths

$dr,\ r\sin\theta\,d\phi,\ r\,d\theta$ vector differential length: $d\mathbf{l} = dr\,\mathbf{a}_r + r\sin\theta\,d\phi\,\mathbf{a}_\phi + r\,d\theta\,\mathbf{a}_\theta$

differential areas

(ds) $r\sin\theta\,d\phi\,dr,\ r^2\sin\theta\,d\phi\,d\theta,\ r\,d\theta\,dr$

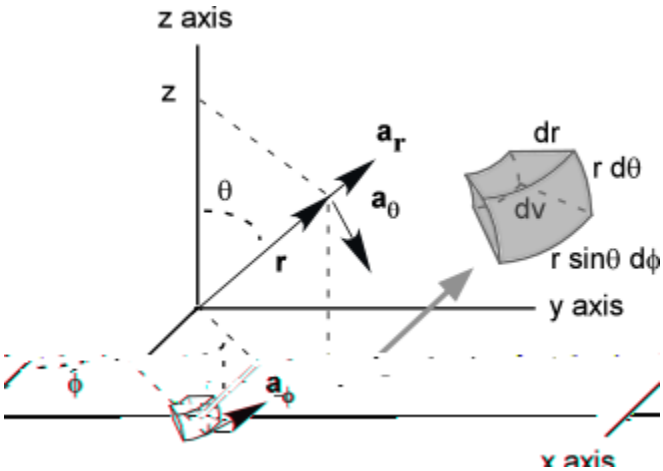
(d**s**) $\pm r\sin\theta\,d\phi\,dr\,\mathbf{a}_\theta,\ \pm r^2\sin\theta\,d\phi\,d\theta\,\mathbf{a}_r,\ \pm r\,d\theta\,dr\,\mathbf{a}_\phi$

differential volume

$r^2\sin\theta\,dr\,d\phi\,d\theta$

variable range

0 to ∞ for r, 0 to π for θ, 0 to 2π for ϕ



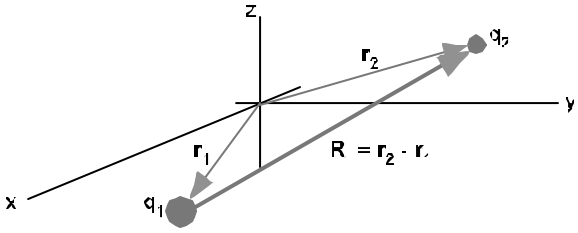
Spherical coordinates are useful in systems with spherical symmetry.

1. None of the unit vectors have a constant direction. The direction of each is a function of position.
2. The unit vectors are mutually orthogonal. $\mathbf{a}_r \cdot \mathbf{a}_\phi = \mathbf{a}_\phi \cdot \mathbf{a}_\theta = \mathbf{a}_r \cdot \mathbf{a}_\theta = 0$.

Electrostatics

In the 18th century, Charles-Augustin de Coulomb found the force between two charges acts on a line connecting them, that it is proportional to the product of their charges, and that it is inversely proportional to the distance between them. The force is attractive if the charges are of opposite sign and repulsive for like signs. The relation is referred to as **Coulomb's law**.

$$\mathbf{F} \propto \frac{q_1 q_2}{|\mathbf{R}|^2}\mathbf{a}_R \quad \text{(force on } q_2 \text{ due to } q_1\text{)}$$



In the MKS system, $\mathbf{F} = \frac{q_1 q_2}{4\pi\varepsilon R^2}\mathbf{a}_R$, where ε is the permittivity of the material between q_1 and q_2.

The permittivity of vacuum, or free space, $\varepsilon_o = 8.854(10^{-12})$ F/m.

Coulomb's law is linear with respect to sources, so that **superposition** holds.

Coulomb's law and superposition: an example

Calculate the force that n charges, q_1 through q_n, at positions $\mathbf{r}_1$ through $\mathbf{r}_n$, exert on charge q at $\mathbf{r}$.

$$\mathbf{F} = \sum_{i=1}^{n} \frac{q_i q}{4\pi\varepsilon R_i^2}\mathbf{a}_i$$

$\mathbf{R}_i = \mathbf{r} - \mathbf{r}_i$ (vector from position of i^{th} charge, $\mathbf{r}_i$, to charge q at $\mathbf{r}$)

$R_i = |\mathbf{R}_i|$ (distance from i^{th} charge to charge q)

$\mathbf{a}_i = \mathbf{R}_i / R_i$ (unit vector from i^{th} charge to charge q)

Electrostatic fields

The electrostatic field, **E**, is the force per unit charge on a test charge q as the charge of the test charge goes to zero.

$$\mathbf{E} = \lim_{q \to 0} \frac{\mathbf{F}}{q}$$

The reason the electric field is defined as a limit is that a finite charge would carry its own field which would affect the very field being characterized by measuring the force on q. That is, if q were finite, its field would affect the **F** being measured. Defining the electric field via an infinitesimal test charge avoids this difficulty.

Electric fields and superposition

From the force exerted on q by one charge q',

$$\mathbf{F} = \frac{q'q}{4\pi\varepsilon R^2}\mathbf{a}_R$$

The corresponding electric field is

$$\mathbf{E} = \lim_{q\to 0}\frac{\mathbf{F}}{q} = \frac{q'}{4\pi\varepsilon R^2}\mathbf{a}_R$$

$\mathbf{R} = \mathbf{r} - \mathbf{r}'$ (vector from position q' at $\mathbf{r}'$ to charge q at $\mathbf{r}$)

$R = |\mathbf{R}|$ (distance from q' to q)

$\mathbf{a}_R = \mathbf{R}/R$ (unit vector from q' to q)

The force exerted on q by a collection of five charges was found above to be

$$\mathbf{F} = \sum_{i=1}^{n}\frac{q_i q}{4\pi\varepsilon R_i^2}\mathbf{a}_i$$

The corresponding electric field is therefore

$$\mathbf{E} = \lim_{q\to 0}\frac{\mathbf{F}}{q} = \sum_{i=1}^{n}\frac{q_i}{4\pi\varepsilon R_i^2}\mathbf{a}_i$$

Continuous charge distributions

The results from the preceding example can be extended to include the field from continuous charge distributions rather than from discrete charges.

$$\mathbf{E} = \int_{q'}\frac{dq'}{4\pi\varepsilon R^2}\mathbf{a}_R$$

For volume charge distributions ($dq' = \rho_v dv'$)

$$\mathbf{E} = \iiint_{\text{charge}}\frac{\rho_v}{4\pi\varepsilon R^2}dv'\,\mathbf{a}_R$$

$dq' = \rho_v dv'$ at $\mathbf{r}'$	infinitesimal source
$\mathbf{r}'$	position of source charge (dq')--source point
$\mathbf{r}$	position of test charge--field point
$\mathbf{R} = \mathbf{r} - \mathbf{r}'$	vector from source to field point
$\mathbf{a}_R = \mathbf{R}/R$	unit vector **from** source **to** field point

Similarly, for surface charge distributions $\mathbf{E} = \iint_{\text{charge}}\frac{\rho_s}{4\pi\varepsilon R^2}ds'\,\mathbf{a}_R$

And, for line charge distributions $\mathbf{E} = \int_{\text{charge}}\frac{\rho_L}{4\pi\varepsilon R^2}dl'\,\mathbf{a}_R$

Charge distributions

Charge always exists in discreet chunks – the magnitude of the charge of an individual electron is $1.6(10^{-19})$ Coulomb.

Since charge occurs in discreet quantities, charge can never be truly distributed. There is always granularity. Uniform distributions are useful approximations when the scale of interest is much larger than the scale in which granularity is evident. These approximations are used constantly in macroscopic electromagnetics and it is well to discuss them at this time.

ρ_v Volume charge distributions (C/m^3) are used for charge distributed over a volume.

ρ_s Surface charge distributions (C/m^2) are used to indicate charge is distributed over a given surface. *This is itself an approximation to what is actually a volume charge density where the charge is distributed in a region very close to some surface. Consider a charged electrical conductor in which its excess charge is confined to a region very close to the surface of the conductor. In this case, it is reasonable to describe the charge distribution by a surface charge distribution (in C/m^2) rather than describe the distribution as a volume charge distribution.*

ρ_L Likewise, line charge distributions (C/m) are used for charge distributed over a length, as, for example, in a charged wire.

Example: infinite line charge ☼

Find the electric field due to a line charge along the z axis.

$$\mathbf{E} = \int_{\text{charge}} \frac{\rho_L}{4\pi\varepsilon R^2} dl' \, \mathbf{a}_R$$

$\mathbf{r}' = z' \, \mathbf{a}_z$ $\qquad$ $\mathbf{r} = \rho \, \mathbf{a}_\rho + z \, \mathbf{a}_z$

$\mathbf{R} = \mathbf{r} - \mathbf{r}' = \rho \, \mathbf{a}_\rho + (z - z') \, \mathbf{a}_z$ $\qquad$ $\mathbf{a}_R = \mathbf{R} / R$

$$\mathbf{E} = \int_{\text{charge}} \frac{\rho_L}{4\pi\varepsilon R^2} dl' \, \mathbf{a}_R = \int_{z'=-\infty}^{\infty} \frac{\rho_L}{4\pi\varepsilon\left[\rho^2 + (z-z')^2\right]} \frac{\rho \mathbf{a}_\rho + (z-z')\mathbf{a}_z}{\sqrt{\left[\rho^2 + (z-z')^2\right]}} dz'$$

$$\mathbf{E} = \int_{z'=-\infty}^{\infty} \frac{\rho_L}{4\pi\varepsilon\left[\rho^2 + (z-z')^2\right]^{\frac{3}{2}}} \left[\rho \mathbf{a}_\rho + (z-z')\mathbf{a}_z\right] dz'$$

Since the limits are from $z' = -\infty$ to ∞, any finite value of z does not affect the result.

$$\mathbf{E} = \int_{z'=-\infty}^{\infty} \frac{\rho_L}{4\pi\varepsilon\left[\rho^2 + z'^2\right]^{\frac{3}{2}}} \left[\rho \mathbf{a}_\rho - z' \mathbf{a}_z\right] dz'$$

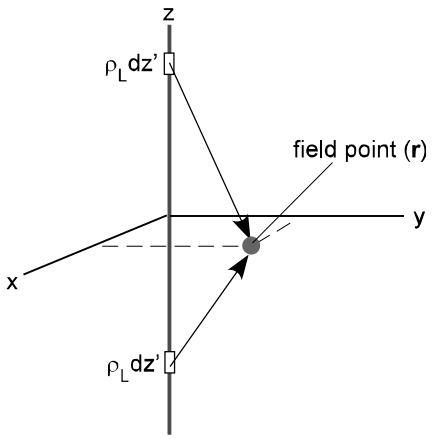
Since the limits along z' are symmetric, the $\mathbf{a}_z$ component is zero after integration.

$$\mathbf{E} = \int_{z'=-\infty}^{\infty} \frac{\rho_L}{4\pi\varepsilon\left[\rho^2 + z'^2\right]^{\frac{3}{2}}} \rho\mathbf{a}_\rho dz'$$

$$\mathbf{E} = \frac{\rho_L \rho\mathbf{a}_\rho}{4\pi\varepsilon} \int_{z'=-\infty}^{\infty} \frac{dz'}{\left[\rho^2 + z'^2\right]^{\frac{3}{2}}}$$

$$\mathbf{E} = \frac{\rho_L \rho\mathbf{a}_\rho}{4\pi\varepsilon} \left(\frac{z'}{\rho^2\sqrt{z'^2 + \rho^2}} \Bigg|_{z'=-\infty}^{\infty} \right)$$

$$\mathbf{E} = \frac{\rho_L \rho\mathbf{a}_\rho}{4\pi\varepsilon} \frac{2}{\rho^2} = \frac{\rho_L}{2\pi\varepsilon\rho}\mathbf{a}_\rho$$



Example: infinite sheet of charge

Consider an infinite plane of charge over the z = 0 plane.

$$= \int_{\text{charge}} \frac{\rho_s}{4\pi\varepsilon R^2} ds' \quad R$$

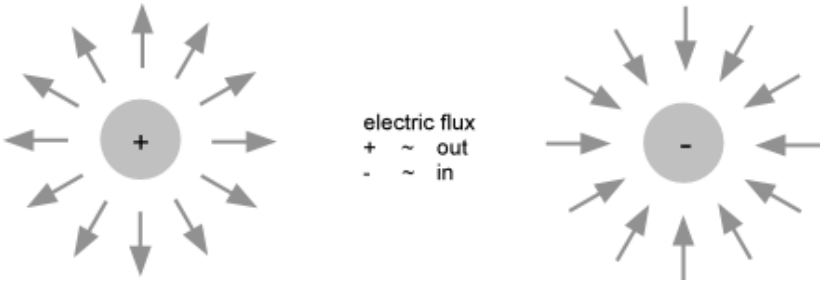
To early researchers, action-at-a-distance relations such as Newton's law of gravitation and Coulomb's law smacked of mysticism or magic. They sought an explanation that would mesh with their experience and be consistent with intuition – in short, they sought mechanical models to describe and think about EM fields.

In the 19th century, Michael Faraday proposed that charges and currents create *electromagnetic fields* which act as an intermediary agent from source to effect. In particular, for electric fields, Faraday pictured flux lines emanating from electric charge.

Gauss' law states that the total electric flux leaving a charge is equal to the charge.

$$\psi_e = q$$

Where ψ_e is the total electric flux coming **out** from charge q.



The total electric flux (in Coulomb) can also be found by taking the surface integral of the electric flux density vector, **D** over a closed surface which encloses the charge q.

Using the flux density, Gauss' law reads $\oiint_{\text{surface}} \mathbf{D} \cdot d\mathbf{s} = q$

The oval in the double integral sign indicates the integral is over a **closed surface**—a surface that separates inside from outside, a boundary with no holes.

In this relation, **D** is the electric flux density vector in C/m^2. The differential surface element, d**s**, is also a vector quantity. Since Gauss' law equates outward flux to charge, the direction of d**s** should be taken as outward from the surface if the integral is to be equal to the total electric flux coming out from the charge.

Surface integrals

Since electromagnetics involves many surface integrals such as

$$\psi_e = \oiint_{\text{surface}} \mathbf{D} \cdot d\mathbf{s},$$

let's take some time to simply explore surface integrals in more detail.

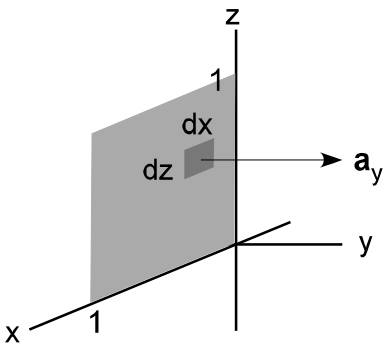
First, note that a vector differential area always has two possible directions for its unit normal. For example dxdy $\mathbf{a}_z$ or dxdy ($-\mathbf{a}_z$).

To find the flux traveling in the direction indicated in the above diagram, the direction for vector differential area is chosen in the same direction. For example, to find the flux leaving a volume, an outward directed differential area (leaving the volume) would be used.

Example: Cartesian

Given $\mathbf{D} = 2xy\,\mathbf{a}_x + 3xz\,\mathbf{a}_y$ C/m^2, find ψ passing through the surface y = 0, $0 \le x \le 1$ m, $0 \le z \le 1$ m in the $\mathbf{a}_y$ direction.

Here, since the flux desired is in the $+\mathbf{a}_y$ direction, d$\mathbf{s}$ is chosen to be dxdz $\mathbf{a}_y$.



$$\mathbf{ds} = dxdz\,\mathbf{a}_y$$

$$\psi = \int_{z=0}^{1}\int_{x=0}^{1}\left[\left(2xy\big|_{y=0}\right)\mathbf{a}_x + 3xz\,\mathbf{a}_y\right] \cdot dxdz\,\mathbf{a}_y$$

$$= 3\left(\frac{x^2}{2}\Big|_0^1\right)\left(\frac{z^2}{2}\Big|_0^1\right) = 0.75 \text{ C}$$

Example: spherical surface

Given $\mathbf{D} = 5/r^2\,\mathbf{a}_r$ C/m^2, find the electric flux passing outward through the surface of a sphere centered at the origin with a radius = 1.5 m.

$$d\mathbf{l} = dr\,\mathbf{a}_r + r\,d\theta\,\mathbf{a}_\theta + r\sin\theta\,d\phi\,\mathbf{a}_\phi$$

The differential length is a good place to start. r is constant on the surface of a sphere centered at the origin, so that $d\mathbf{s} = r\,d\theta\; r\sin\theta\,d\phi\,\mathbf{a}_r = r^2\sin\theta\,d\theta\,d\phi\,\mathbf{a}_r$

$$\psi = \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\left.\frac{5}{r^2}\right|_{r=1.5\text{ m}}\mathbf{a}_r \cdot \left.r^2\right|_{r=1.5\text{ m}}\sin\theta\,d\theta\,d\phi\,\mathbf{a}_r$$

$$= 5\left(-\cos\theta\big|_{\theta=0}^{\pi}\right)\left(\phi\big|_{\phi=0}^{2\pi}\right) = 5(4\pi) \text{ C}$$

Example: spherical surface

Given $\mathbf{D} = 5/r^2 \sin\theta\ \mathbf{a}_r$ C/m^2, find the electric flux passing outward through the surface of a sphere centered at the origin with a radius = 1.5 m.

$$\psi = \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \left.\frac{5}{r^2}\right|_{r=1.5\text{ m}} \sin\theta\ \mathbf{a}_r \bullet \left. r^2\right|_{r=1.5\text{ m}} \sin\theta\, d\theta\, d\phi\ \mathbf{a}_r$$

$$= 5\left(\left.\phi\right|_{\phi=0}^{2\pi}\right)\int_{\theta=0}^{\pi}\sin^2\theta\, d\theta = 5(2\pi)\int_{\theta=0}^{\pi}\left(\frac{1}{2} - \frac{\cos 2\theta}{2}\right)d\theta$$

$$= 5\pi^2 \text{ C}$$

Example: cylindrical surface

Given $\mathbf{D} = 5/\rho\ \mathbf{a}_\rho$ C/m^2, find the total electric flux passing outward through a closed cylindrical surface with its axis on the z-axis, with radius 4 cm, extending from $-3 \text{ cm} \le z \le 2 \text{ cm}$

Step 1: determine differential areas directed outward from surface

sides $\quad d\mathbf{s} = \left.\rho\right|_{\rho=0.04\text{ m}} d\phi\, dz\, \mathbf{a}_\rho = 0.04\, d\phi\, dz\, \mathbf{a}_\rho$

top $\quad d\mathbf{s} = \rho\, d\phi\, d\rho\, \mathbf{a}_z$

bottom $\quad d\mathbf{s} = \rho\, d\phi\, d\rho\, (-\mathbf{a}_z)$

Step 2: form dot product $\mathbf{D} \bullet d\mathbf{s}$ on each surface

sides $\quad \mathbf{D} \bullet d\mathbf{s} = \left.\frac{5}{\rho}\right|_{\rho=0.04\text{m}} \mathbf{a}_\rho \bullet 0.04\, d\phi\, dz\, \mathbf{a}_\rho = 5\, d\phi\, dz$

top and bottom, $\mathbf{D} \bullet d\mathbf{s} = 0$

Step 3: determine limits and perform the integral $\psi = \oiint_{\text{surface}} \mathbf{D} \bullet d\mathbf{s}$

answer: $\psi = 0.5\pi$ C

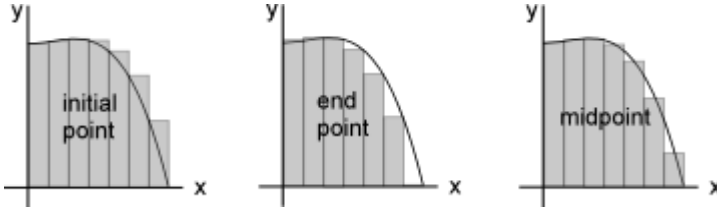
Numeric Approximation

One classic means of estimating the area of a definite integral is to express the integral as a **Riemann sum** and approximate the area under the curve as the sum of the rectangular areas of constant width and whose heights are determined by the function's value.

One possibility in determining rectangle heights would be to use the value of the function at the initial point in the interval as the height of the rectangle. This would lead to the formation of a **left** Riemann sum. Another possibility would be to use the value at the end point for a **right** Riemann sum, and another would be to use the value at the midpoint for a **middle** Riemann sum. Here, we will use a middle Riemann sum.

The illustration below compares these approaches to approximating an integral. The first uses rectangle heights at the initial point of an interval (width of rectangle), the second uses the end point, and the third uses the value at the midpoint of the interval.

Taking the value at the midpoint of an interval results in a robust approximation that is more accurate for most curves.



Numerical approximation: finite differences

Given $\mathbf{D} = 2x\,\mathbf{a}_x + 3x^3y\,\mathbf{a}_z$ C/m^2, find the approximate electric flux, ψ, passing through the surface $z = 0,\ 0 \le x \le 1$ m, $0 \le y \le 1$ m in the $\mathbf{a}_z$ direction. Use (0.1 m)^2 areas.

It may be more straightforward to form the sum from the integral.

$$\int_{y=0}^{1}\int_{x=0}^{1}\left(2x\,\mathbf{a}_x + 3x^3y\,\mathbf{a}_z\right)\cdot dx\,dy\,\mathbf{a}_z = \int_{y=0}^{1}\int_{x=0}^{1} 3x^3y\,dx\,dy$$

$$\int_{y=0}^{1}\int_{x=0}^{1}\left(2x\,\mathbf{a}_x + 3x^3y\,\mathbf{a}_z\right)\cdot dx\,dy\,\mathbf{a}_z = \sum_{i=1}^{10}\sum_{j=1}^{10} 3(0.1j - 0.05)^3(0.1i - 0.05)(0.1)(0.1)$$

Using a computer to calculate the sum *(here Maple is used)*

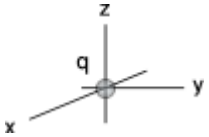
```
>   flux := sum (sum (3 * (0.1 * j - 0.05)^3 * (0.1 * i - 0.05) * 0.1 * 0.1, i = 1..10),j = 1..10);
    flux := .3731250000
```

Example: electric flux density from charge q at the origin.

Consider a point charge q at the origin. From the charge's symmetry, the field must have spherical symmetry, where the magnitude of the electric flux density vector is independent of ϕ or θ and can only depend on r. Symmetry also causes the direction of the electric flux density vector to be in the $\mathbf{a}_r$ direction.

$\mathbf{D} = D\,\mathbf{a}_r$

D **not** a function of ϕ or θ.



Given this, a surface, a **Gaussian surface**, is chosen so that the Gaussian integral can be evaluated. Here, the Gaussian surface is a sphere, centered on the origin.

$$d\mathbf{s} = r^2 \sin\theta\, d\theta d\phi\, \mathbf{a}_r \quad \rightarrow \quad \oiint_{\text{surface}} \mathbf{D}\cdot d\mathbf{s} = \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} D\,\mathbf{a}_r \cdot r^2 \sin\theta\, d\theta d\phi\, \mathbf{a}_r$$

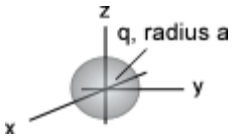
$$Dr^2 \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \sin\theta\, d\theta d\phi = Dr^2 \left(\phi\Big|_0^{2\pi}\right)\left(-\cos\theta\Big|_{\theta=0}^{\pi}\right) = 4\pi Dr^2 = q \quad \rightarrow \quad \mathbf{D} = \frac{q}{4\pi r^2}\mathbf{a}_r$$

Example: spherically symmetric distribution

Consider a volume charge density where a charge q is evenly distributed over a sphere of radius a. Given this charge, the field has spherical symmetry, again meaning that the magnitude of the electric flux density vector must be independent of ϕ or θ and can only depend on r with its direction in the $\mathbf{a}_r$ direction.

$\mathbf{D} = D\,\mathbf{a}_r$

D is **not** a function of ϕ or θ.



The Gaussian surface is a sphere, centered on the origin. The difference here is that there are two cases, one for r<a and the other for r>a.

r<a

$$\oiint_{\text{surface}} \mathbf{D} \cdot d\mathbf{s} = q$$

$$\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} D\,\mathbf{a}_r \cdot r^2 \sin\theta\, d\theta d\phi\, \mathbf{a}_r = \iiint_{\text{volume}} \rho_v\, dv = \int_{\phi'=0}^{2\pi}\int_{\theta'=0}^{\pi}\int_{r'=0}^{r} \frac{q}{\frac{4}{3}\pi a^3} r'^2 \sin\theta'\, dr' d\theta' d\phi'$$

$$Dr^2 \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \sin\theta\, d\theta d\phi = 4\pi D r^2 = \frac{q}{\frac{4}{3}\pi a^3}\frac{4\pi}{3} r^3$$

$$\mathbf{D} = \frac{q}{4\pi r^2}\frac{r^3}{a^3}\mathbf{a}_r = \frac{qr}{4\pi a^3}\mathbf{a}_r$$

r>a

$$\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} D\,\mathbf{a}_r \cdot r^2 \sin\theta\, d\theta d\phi\, \mathbf{a}_r = \iiint_{\text{volume}} \rho_v\, dv = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r'=0}^{a} \frac{q}{\frac{4}{3}\pi a^3} r'^2 \sin\theta'\, dr' d\theta' d\phi'$$

$$Dr^2 \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \sin\theta\, d\theta d\phi = 4\pi D r^2 = \frac{q}{\frac{4}{3}\pi a^3}\frac{4\pi}{3} a^3$$

$$\mathbf{D} = \frac{q}{4\pi r^2}\mathbf{a}_r$$

Notice the only difference in the mathematical procedure between the two cases is in the r limit in the volume integral which changes from a (for the r<a case) to r (r>a case).

This one change results in marked differences in field behavior.

Electric flux density

The electric flux density vector is related to the electric field via permittivity. The electric field due to a point charge q at the origin (force per unit charge from Coulomb's law).

$$\mathbf{E} = \frac{q}{4\pi\varepsilon r^2}\mathbf{a}_r$$

The electric flux density due to a point charge q at the origin (from Gauss' law)

$$\mathbf{D} = \frac{q}{4\pi r^2}\mathbf{a}_r$$

Therefore $\mathbf{D} = \varepsilon\mathbf{E}$. This relation is general for linear homogenous materials, not just for point charges at the origin—more about permittivity later.

Example: line of charge ☼

Use Gauss' law to determine the field due an infinite line of charge, ρ_L, on the z-axis. Due to the clear symmetry involved, cylindrical coordinates will be used.

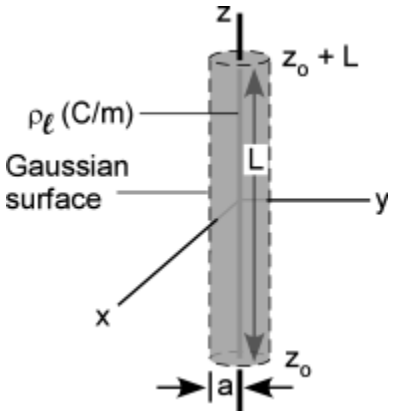
The fact that the charge is an infinite line charge along the z-axis permits the form of the field to be determined: 1) since the line charge is infinite in extent along z, the strength of the field cannot depend on z and neither can the field have a component in the $\mathbf{a}_z$ direction, 2) since the charge producing the field is an infinite line charge, symmetry implies that the strength of the field cannot depend on ϕ and neither can the field have a component in the $\mathbf{a}_\phi$ direction, and 3) these considerations require the form of the field to be $\mathbf{D} = D\,\mathbf{a}_\rho$, where D can only depend on ρ.

The field's symmetry is the key in using Gauss' law to determine the field. With this symmetry, a Gaussian surface (a cylinder with its axis on the z-axis. Choose the cylinder to have radius **a** and length L as shown on the diagram below) can be chosen such that the surface integral (the Gaussian integral) becomes tractable.

Note the central role of symmetry in this problem:

1. The field cannot depend on z since the line charge is on the z axis and the line is considered infinite in length. The physical environment is not a function of z, so neither is the field.
2. The field cannot depend on ϕ since the line charge is on the z axis which so that the physics of the situation is also independent of ϕ.
3. The field can only depend on ρ and, moreover, must point in the $\pm\mathbf{a}_\rho$ direction, depending on whether ρ_L is positive or negative.

Notice how the Gaussian surface serves to make the evaluation of the Gaussian integral simpler.



1) The field is perpendicular to the surface normal on the end of the cylinder.

2) The field is parallel to the surface normal on the cylinder walls. Moreover, the field must be a constant on the cylinder walls since ρ is constant and it is established that D = D(ρ).

$$\oiint \mathbf{D} \cdot d\mathbf{s} = q$$

$$\iint_{\text{sides}} \mathbf{D} \cdot d\mathbf{s} + \iint_{\text{top}} \mathbf{D} \cdot d\mathbf{s} + \iint_{\text{sides}} \mathbf{D} \cdot d\mathbf{s} = q_{\text{inside}}$$

$$\int_{z=z_o}^{z_o+L} \int_{\phi=0}^{2\pi} D\,\mathbf{a}_\rho \cdot \rho\, d\phi dz\, \mathbf{a}_\rho + \int_{\rho=0}^{a} \int_{\phi=0}^{2\pi} D\,\mathbf{a}_\rho \cdot \rho\, d\phi d\rho\, \mathbf{a}_z + \int_{\rho=0}^{a} \int_{\phi=0}^{2\pi} D\,\mathbf{a}_\rho \cdot \rho\, d\phi d\rho\, (-\mathbf{a}_z) = \rho_l\, L$$

$$\int_{z=z_o}^{z_o+L} \int_{\phi=0}^{2\pi} D\,\rho\, d\phi dz = (D\,\rho) \int_{z=z_o}^{z_o+L} \int_{\phi=0}^{2\pi} d\phi dz = \rho_l\, L$$

$$D\,\rho\, L\, 2\pi = \rho_l\, L \quad \rightarrow \quad D = \frac{\rho_l}{2\pi\rho}$$

Knowing the magnitude and recalling from symmetry arguments that $\mathbf{D} = D\,\mathbf{a}_\rho$.

$$\mathbf{D} = \frac{\rho_L}{2\pi\rho}\,\mathbf{a}_\rho \quad \rightarrow \quad \mathbf{E} = \frac{\rho_L}{2\pi\varepsilon\rho}\,\mathbf{a}_\rho$$

Example: infinite cylinder with its surface charged

Determine the electric field for a cylinder with a surface charge, ρ_s = 3 pC/m^2. The cylinder has the z-axis as its axis. The radius of the cylinder, ρ = 0.15 m. The cylinder is infinite in extent, -∞< z < ’.

$$\rho_s = \begin{cases} 3 \text{ pC/m}^2 & \rho = 0.15 \text{ m} \\ 0 & \rho \neq 0.15 \text{ m} \end{cases}$$

There is cylindrical symmetry with $\rho_L = 2\pi\,(0.15\text{ m})\,\rho_s$. Using the above result, one obtains.

$$\text{answer:}\quad \mathbf{E} = \begin{cases} 0 & \rho < 0.15 \text{ m} \\ \dfrac{0.15(2\pi)3(10^{-12})}{2\pi\varepsilon\rho}\,\mathbf{a}_\rho \ \dfrac{\text{N}}{\text{C}} & \rho > 0.15 \text{ m} \end{cases}$$

Example: distributed charge

Determine the flux density from an infinitely long cylinder of radius a with a distributed volume charge density, find **D** for ρ < a and for ρ > a.

$$\rho_v = \begin{cases} 2\rho \ \text{C/m}^3 & \rho < a \\ 0 & \rho > a \end{cases}$$

From the charge distribution, it is clear the field has cylindrical symmetry so that the Gaussian surface is a cylinder of length L with its axis along the z-axis.

$$\oiint \mathbf{D} \cdot d\mathbf{s} = q_{inside} = \begin{cases} \displaystyle\int_{z=z_o}^{z_o+L}\int_{\phi=0}^{2\pi}\int_{\rho=0}^{\rho} 2\rho\ (\rho d\rho d\phi dz) & \rho < a \\ \displaystyle\int_{z=z_o}^{z_o+L}\int_{\phi=0}^{2\pi}\int_{\rho=0}^{a} 2\rho\ (\rho d\rho d\phi dz) + \int_{z=z_o}^{z_o+L}\int_{\phi=0}^{2\pi}\int_{\rho=a}^{\rho} 0\ (\rho d\rho d\phi dz) & \rho > a \end{cases}$$

$$\text{answer:}\quad \mathbf{D} = \begin{cases} \dfrac{2\rho^2}{3}\,\mathbf{a}_\rho & \rho < a \\ \dfrac{2a^3}{3\rho}\,\mathbf{a}_\rho & \rho > a \end{cases}$$

Example: Coulomb's law implies Gauss' law

For a charge q at the origin, the force on a test charge q_t is

$$\mathbf{F} = \frac{q q_t}{4\pi\varepsilon r^2}\mathbf{a}_r$$

The electric field—the force per unit charge due to the charge q at the origin is

$$\mathbf{E} = \frac{q}{4\pi\varepsilon r^2}\mathbf{a}_r$$

Integrating **E** over a sphere, centered on the origin, surrounding the charge q and using the definition of permittivity, $\mathbf{D} = \varepsilon\mathbf{E}$.

$$\oiint_{\text{sphere}} \mathbf{D}\cdot d\mathbf{s} = \oiint_{\text{sphere}} \frac{q}{4\pi r^2}\mathbf{a}_r \cdot d\mathbf{s} = \oiint_{\text{sphere}} \frac{q}{4\pi r^2}\mathbf{a}_r \cdot r^2 \sin\theta \, d\phi \, d\theta \, \mathbf{a}_r = q$$

$$\oiint_{\text{sphere}} \mathbf{D}\cdot d\mathbf{s} = q$$

So that using Gauss' law shows the charge to be q as was derived from Coulomb's law.

The importance of symmetry in analytic solutions

In the two preceding techniques—using superposition integrals and using Gauss' law, the ability to find analytic solutions depended upon the charge distribution's symmetry.

First, when using Coulomb's law in the superposition integral, the charge's symmetry allowed us to analytically evaluate the resulting integrals to obtain a closed-form solution. It is true that the integrals could have readily been written without the given symmetry since only a knowledge of the charge's distribution in space is needed for this. But, for analytic evaluation, symmetry was a critical factor.

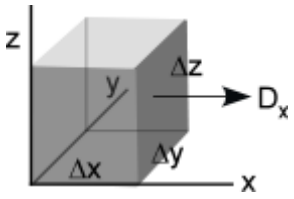
Similarly, when using Gauss' law, symmetry often resulted in the existance of a suitable Gaussian surface. This allowed the crucial step of evaluating the Gaussian integral to be carried out—often not possible without sufficient symmetry. With symmetry and the proper choice of Gaussian surface, evaluating the Gaussian integral is often trivial. Without this symmetry, however, its evaluation would often be anything but trivial and usually required numeric evaluation.

How does one approach a more general situation, where perhaps the charge is not so conveniently arranged? One approach is to use a numeric tool like MATLAB to evaluate the resulting superposition integrals—an approach which will be explored via homework.

This approach is general for those cases in which the charge distribution is known. Unfortunately, in many cases, we're faced with a boundary value problem in which the fields or potential on a boundary surface is known and the charge is an unknown. In this case the knowns include the region's geometry, material properties, and boundary conditions. We have, therefore, two types of problems—one in which we know the charge and the other in which we know boundary conditions. Both types are important. Boundary value problems are solved via differential equations and, therefore, a mathematical description of electromagnetics in terms of differential equations is important.

Divergence

Consider a cube of dimension $\Delta x\ \Delta y\ \Delta z$ with one corner at xyz.



The net flux leaving the volume $\Delta x\ \Delta y\ \Delta z$ in the $\mathbf{a}_x$ direction is:

$$\psi_{e\text{-}x\ (out)} = [D_x(x + \Delta x, y, z) - D_x(x,y,z)](\Delta y\ \Delta z)$$

Divide and multiply by Δx

$$\psi_{e\text{-}x\ (out)} = \left\{\frac{[D_x(x + \Delta x, y, z) - D_x(x,y,z)]}{\Delta x}\right\} (\Delta x\ \Delta y\ \Delta z)$$

In this expression, if Δx, Δy, and $\Delta z \rightarrow 0$, the result is:

$$(d\psi_{e\text{-}out})_x = \frac{\partial D_x}{\partial x} dx\ dy\ dz = \frac{\partial D_x}{\partial x}\ dv$$

Similarly,

$$(d\psi_{e\text{-}out})_y = \frac{\partial D_y}{\partial y}\ dv \qquad \text{and} \qquad (d\psi_{e\text{-}out})_z = \frac{\partial D_z}{\partial z}\ dv$$

$$d\psi_{e\text{-}out} = (d\psi_{e\text{-}out})_x + (d\psi_{e\text{-}out})_y + (d\psi_{e\text{-}out})_z$$

$$= \frac{\partial D_x}{\partial x} dv + \frac{\partial D_y}{\partial y} dv + \frac{\partial D_z}{\partial z} dv = (\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z})\ dv$$

The quantity within the parentheses occurs often and it called the divergence of **D**. div **D** can be written as $\nabla \bullet \mathbf{D}$, where ∇ is the "del" operator

$$\nabla = \frac{\partial}{\partial x}\mathbf{a}_x + \frac{\partial}{\partial y}\mathbf{a}_y + \frac{\partial}{\partial z}\mathbf{a}_z$$

The divergence is formed by the dot product of the del operator and a vector, in this case the current density.

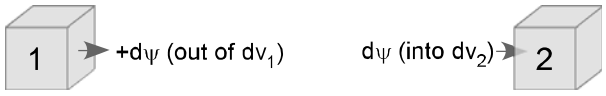
$$\nabla \bullet \mathbf{D} = \left(\frac{\partial}{\partial x}\mathbf{a}_x + \frac{\partial}{\partial y}\mathbf{a}_y + \frac{\partial}{\partial z}\mathbf{a}_z \right) \bullet \left(D_x\, \mathbf{a}_x + D_y\, \mathbf{a}_y + D_z\, \mathbf{a}_z \right)$$

$$\nabla \bullet \mathbf{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$

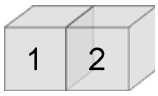
The divergence of **D**, the electric flux density vector, is the net electric flux out per unit volume.

$$d\psi_{e\text{-}out} = \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right) dv$$

Can this differential relation be used to find the electric flux out of macroscopic bodies? To answer this question, consider two adjacent differential cubes. The critical question whether the electric flux at the adjacent areas is properly accounted for to allow integration over macroscopic volumes



It is apparent that, when taking the divergence over adjacent differential volumes, the common flux across their shared surface areas will be positive for one and negative which will cancel for the composite volume.



This observation for two adjacent differential volumes can be extended to any number of differential volumes. This permits the use of integration of an infinite number of differential volumes for the case of macroscopic bodies and allows divergence to be used to find the electric flux out of macroscopic volumes.

$$\psi_{e\text{-}out} = \iiint_{volume} \nabla \bullet \mathbf{D}\, dv$$

Example: charge within volume

Given that the electric field is $\ddot{u} = \frac{75}{4\pi\varepsilon\sqrt{r}}\,\mathbf{a}_r\ \frac{N}{C}$, find the total charge within a sphere of radius 10 cm which is centered on the origin. Do this in two ways: with a surface integral and then with a volume integral.

1) surface integral

by Gauss' law $q = \psi_e = \oiint_{surface} \mathbf{D} \cdot d\mathbf{s}$

$$\ddot{u} = \frac{75}{4\pi\varepsilon\sqrt{r}}\,\mathbf{a}_r\ \frac{N}{C} \quad \rightarrow \quad \mathbf{D} = \frac{75}{4\pi\sqrt{r}}\,\mathbf{a}_r\ \frac{C}{m^2}$$

$$\psi_e = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} \frac{75}{4\pi\sqrt{r}}\,\mathbf{a}_r \cdot r^2 \sin\theta\, d\theta\, d\phi\, \mathbf{a}_r \Big|_{r=0.1\ m} = 2.37\ C$$

2) volume integral

$$q = \psi_e = \oiint_{surface} \mathbf{D} \cdot d\mathbf{s} = \iiint_{volume} \nabla \cdot \mathbf{D}\, dv$$

$$\nabla \cdot \mathbf{D} = \frac{1}{r^2}\frac{\partial\left(r^2 \frac{75}{4\pi\sqrt{r}}\right)}{\partial r}\ \frac{C}{m^3} = \frac{1}{r^2}\left(\frac{3}{2}r^{\frac{1}{2}}\right)\frac{75}{4\pi}\ \frac{C}{m^3} = \frac{3}{2r^{3/2}}\frac{75}{4\pi}\ \frac{C}{m^3}$$

$$\iiint_{volume} \nabla \cdot \mathbf{D}\, dv = \int_{r=0}^{0.1}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} \frac{3}{2r^{3/2}}\frac{75}{4\pi}\, r^2 \sin\theta\, d\theta\, d\varphi\, dr = 2.37\ C$$

Divergence Theorem

Since the net electric flux out for a macroscopic volume can also be expressed in terms of a surface integral, the result is a relation between a volume integral and a surface integral.

$$\psi_{e\text{-}out} = \iiint_{volume} \nabla \cdot \mathbf{D}\, dv = \oiint_{surface} \mathbf{D} \cdot d\mathbf{s}$$

Although this relation has been developed for the case of electric flux, the derivation has been purely mathematical and holds for any vector quantity. The relation is known as the **divergence theorem**.

If the volume is reduced to dv, it can be seen that the physical meaning of $\nabla \cdot \mathbf{D}$ is the net electric flux out per unit volume.

What is this quantity, the net electric flux out per unit volume? Using Gauss' law which states the electric flux out of a charge is equal to its charge, the flux out per unit volume must be the charge per unit volume, ρ_v. The result is the point form of Gauss' law which is discussed further below.

Point form of Gauss' law

Combining the divergence theorem for electric flux density with Gauss' law,

$$\psi_{e\text{-out}} = q$$

$$\iiint_{volume} \nabla \cdot \mathbf{D}\, dv = \iiint_{volume} \rho_v\, dv$$

This relation holds for *any* volume. The only possible way for two volume integrals to be equal for an arbitrary volume is for the integrands themselves to be equal. The result is the **point form of Gauss' law.**

$$\nabla \cdot \mathbf{D} = \rho_v$$

From the meaning of divergence of a flux density (net flux out per unit volume), $\nabla \cdot \mathbf{D}$ must be the electric flux out per unit volume. Taking this observation together with Gauss' law, which states that the electric flux coming from a volume is equal to the net charge within the volume, the electric flux out per unit volume can be thought of as the charge per unit volume. This is the meaning of the point form of Gauss' law.

In terms of the electric field.

$$\nabla \cdot \varepsilon \mathbf{E} = \rho_v$$

If the permittivity is not a function of position, it passes unchanged through the del operator.

$$\nabla \cdot \mathbf{E} = \frac{\rho_v}{\varepsilon}$$

So far, the focus of discussion has been on coordinate systems, vectors, vector calculus and the relationship between charge and the electric field, electric flux and electric flux density. Now, electromagnetics will be viewed through the lens of energy and potential. One advantage in looking at the problem in this way is that one deals with scalar functions (energy and potential), which can then be used to find fields.

Potential Energy

A charge q in an electrostatic field **E**, is acted upon by the field with a force **F** = q**E**. To move charge q against this field, an equal and opposite force, $F_{applied} = -\mathbf{F}$, must be applied to move charge.

As the charge moves, work is done on the charge by the applied force. This work is not lost to thermal energy through friction. The work done on the charge acts to increase the charge's potential energy, just as pushing against a spring increases the spring's potential or just as moving a ball uphill in a gravitational field increases its potential energy.

The reference point for potential energy—the point where the potential energy is considered to be zero—is arbitrary. Only **differences** of potential energy are physically meaningful (just as in circuit analysis where the absolute potential of the reference node is arbitrary, since only potential differences affect circuit behavior). In electromagnetics, the reference is typically taken to be at infinity. Also, by convention, the potential at infinity is take to be zero.

As charge is moved from infinity to a position **r**, a force must be applied to the charge which precisely balances the force exerted on the charge by the electric field. In moving the charge, this force does work on the charge. The differential of this potential energy is

$$dW = \mathbf{F}_{applied} \bullet d\mathbf{l} = -q\mathbf{E} \bullet d\mathbf{l}$$

The resulting electric potential energy of the charge q at position **r** in the field **E** is

$$\int_{W=0}^{W} dW' = \int_{\infty}^{\mathbf{r}} -q\mathbf{E} \bullet d\mathbf{l}$$

$$W(\mathbf{r}) = \int_{\infty}^{\mathbf{r}} -q\mathbf{E} \bullet d\mathbf{l}$$

Rather than tracking the potential energy of a particular charge q, the potential energy per charge, the **electrostatic potential**, is often used.

$$V(\mathbf{r}) = \frac{W(\mathbf{r})}{q} = \int_{\infty}^{\mathbf{r}} -\mathbf{E} \bullet d\mathbf{l}$$

The electrostatic potential is analogous to a node voltage in circuit analysis. In nodal analysis, a node voltage is defined as having its positive at the node in question and having its negative sign at the reference node. Here, the electrostatic potential is referenced to infinity. Infinity is the "reference node" for electrostatic potential, the place where the negative sign goes.

Example—potential difference between points a **and** b. ☼

$$V(a) = \frac{W(a)}{q} = \int_{\infty}^{a} -\mathbf{E}\cdot d\mathbf{l} \qquad V(b) = \frac{W(b)}{q} = \int_{\infty}^{b} -\mathbf{E}\cdot d\mathbf{l}$$

$$V_{ab} = V(a) - V(b) = \int_{\infty}^{a} -\mathbf{E}\cdot d\mathbf{l} - \int_{\infty}^{b} -\mathbf{E}\cdot d\mathbf{l} = \int_{b}^{a} -\mathbf{E}\cdot d\mathbf{l}$$

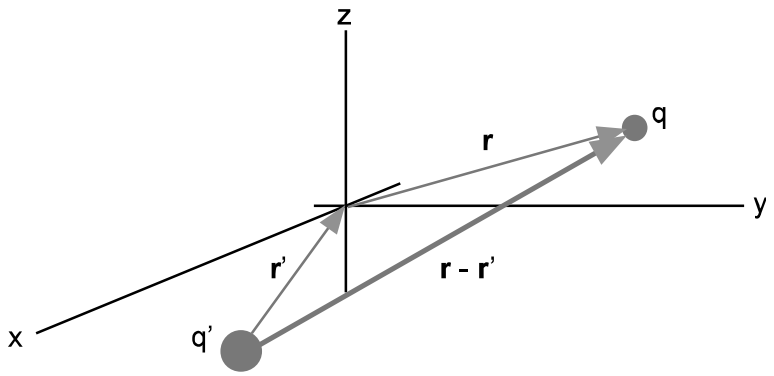
$$V_{ab} = \int_{b}^{a} -\mathbf{E}\cdot d\mathbf{l}$$

Notice that is the work per unit charge expended in moving charge from **b** to **a** *against* the field **E**. The following formula might be a convenient form in which to remember potential differences.

$$V_{ab} = \int_{a}^{b} \mathbf{E}\cdot d\mathbf{l} \qquad V_{+-} = \int_{+}^{-} \mathbf{E}\cdot d\mathbf{l}$$

Example—electrostatic energy and electrostatic potential

Coulomb's law gives the force between charges. The force **F** on charge q at **r** due to charge q' at **r**' is



$$\mathbf{F} = \frac{q\,q'}{4\pi\varepsilon\,|\mathbf{r}-\mathbf{r}'|^2}\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}$$

$|\mathbf{r}-\mathbf{r}'|$ ~ distance between charges

$\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}$ ~ unit vector pointing **from** q' **to** q

Take r' = 0 (q' at the origin).

$$\mathbf{F} = \frac{q\,q'}{4\pi\varepsilon\,|\mathbf{r}|^2}\frac{\mathbf{r}}{|\mathbf{r}|} = \frac{q\,q'}{4\pi\varepsilon\,r^2}\mathbf{a}_r$$

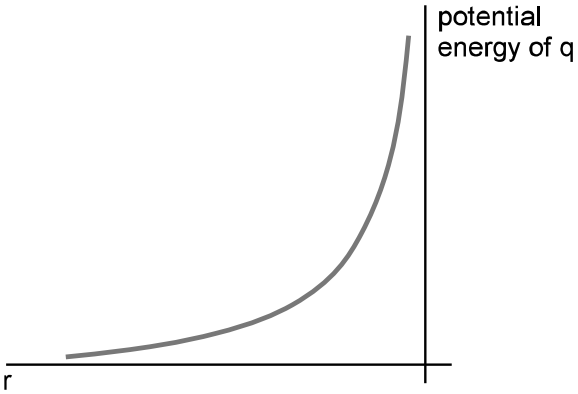
Find the work or energy required to move q at ∞ to an arbitrary position **r**.

As an aside, note the problem has spherical symmetry, which makes the spherical coordinate system the natural coordinate system to use. The applied force -**F** moves q *against* the coulomb force **F**.

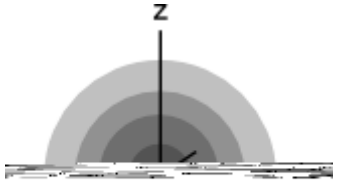
dW = -**F** • d**l** (this is the differential of work done by the applied force -**F**)

$$W(r) = \int_{r=\infty}^{r} -\frac{q\,q'}{4\pi\varepsilon\, r^2}\,dr = -\frac{q\,q'}{4\pi\varepsilon}\left(-\frac{1}{r}\Big|_{\infty}^{r}\right) = \frac{q\,q'}{4\pi\varepsilon\, r}$$

W is the work done on the charge q in moving it from ∞ to the position **r**. The work required to move q is transformed in to potential energy, **electrostatic potential energy**. As the applied force, -**F**, pushes q against the coulomb field established by q', the potential energy of q increases. The charge is being rolled up a potential energy hill. The applied force -**F** pushes q up an electrostatic potential energy hill. The charge moves in mechanical equilibrium (the external force, -**F**, just balancing the field's force, **F**) so that kinetic energy is not increasing. One speaks of work being done "against" the field.



What shape are the surfaces of equipotential energy in this case? Due to the spherical symmetry the field (which results in **F** having a radial direction), the surfaces of equipotential are concentric spheres.



Consider the force required to move q in the coulomb field of q'. What force is required to move q on the surfaces of equipotential energy?

Answer: None, otherwise work would be required for them to move on the surfaces, which would then not be surfaces of equipotential energy.

Potential energy, force, potential, electric field

From mechanics, the differential of energy done by a force -**F** is the dot product of the force and the differential length (dW = -**F** • d**l**). Consider **F** to be the force exerted by the field on a charge q and that -**F** is the force that must be exerted to move the charge **against** the field. Therefore dW = -**F** • d**l** is the differential of the work done by the force –**F** on the charge in moving the charge against the force of the field, **F**. The work done on the charge by the applied or external force, -**F**, is equal to the increase in the potential energy, W, of the charge q.

Now, consider the differential of work from purely a mathematical point-of-view. If the scalar function W is a function of position (of x, y, and z in Cartesian coordinates), then the total differential must be, by the chain rule.

$$dW = \frac{\partial W}{\partial x}dx + \frac{\partial W}{\partial y}dy + \frac{\partial W}{\partial z}dz$$

This quantity can, in turn be expressed as the dot product of the gradient of W and the differential length, d**l**.

$$dW = \frac{\partial W}{\partial x}dx + \frac{\partial W}{\partial y}dy + \frac{\partial W}{\partial z}dz = \left(\frac{\partial W}{\partial x}\mathbf{a}_x + \frac{\partial W}{\partial y}\mathbf{a}_y + \frac{\partial W}{\partial z}\mathbf{a}_z\right) \cdot \left(dx\,\mathbf{a}_x + dy\,\mathbf{a}_y + dz\,\mathbf{a}_z\right)$$

With this observation, one can clearly appreciate the meaning of the first term in the dot product, dW = -**F** • d**l**.

$$-\mathbf{F} = \frac{\partial W}{\partial x}\mathbf{a}_x + \frac{\partial W}{\partial y}\mathbf{a}_y + \frac{\partial W}{\partial z}\mathbf{a}_z \rightarrow \qquad \mathbf{F} = -\left(\frac{\partial W}{\partial x}\mathbf{a}_x + \frac{\partial W}{\partial y}\mathbf{a}_y + \frac{\partial W}{\partial z}\mathbf{a}_z\right)$$

The expression

$$\frac{\partial W}{\partial x}\mathbf{a}_x + \frac{\partial W}{\partial y}\mathbf{a}_y + \frac{\partial W}{\partial z}\mathbf{a}_z$$

is the result of the del operator acting on the scalar function W, called the **gradient** of W.

$$\nabla W = \left(\mathbf{a}_x\frac{\partial}{\partial x} + \mathbf{a}_y\frac{\partial}{\partial y} + \mathbf{a}_z\frac{\partial}{\partial z}\right) W = \frac{\partial W}{\partial x}\mathbf{a}_x + \frac{\partial W}{\partial y}\mathbf{a}_y + \frac{\partial W}{\partial z}\mathbf{a}_z$$

The gradient of W is a vector valued function. It points directly “uphill”, in the direction of the maximum increase in W. Its magnitude is the value of this maximum increase.

The gradient can be taken of other scalar functions of position and the meaning is analogous. The gradient points in the direction of maximum increase in the scalar function and its magnitude is maximum increase.

1. The gradient of a scalar function of position is a vector.
2. The gradient of a scalar function has as its magnitude the maximum rate of increase in the scalar function, and its direction toward maximum increase.

Here the scalar function of position is W, the electrostatic potential energy. What is the meaning of increasing W or decreasing W? It simply refers to the fact that when W increases, the potential energy of the charge q increases. As W decreases, the potential energy of the charge decreases.

What does the gradient of scalar potential energy signify physically? The gradient of W would give the magnitude and direction of the maximum increase in potential energy. It is analogous to the hiker on the side of a hill. The gradient of the gravitational potential energy in this case would indicate "uphill." The gradient of the electrostatic energy is analogous; it also indicates "uphill", this time the hill being an electrostatic potential hill.

The force that the field exerts is downhill to lower potential energies. Greater energies are only reached if an external force does work on the charge in moving it "uphill" to greater potential energy. The force a charge experiences in an electrostatic field is downhill, in the direction of decreasing potential energy.

The force exerted by the electrostatic field on a positive charge (the force $\mathbf{F} = q\mathbf{E}$) points "downhill". To move the charge uphill to greater potential energy, an external force equal to negative the field's force ($-\mathbf{F} = -q\mathbf{E}$) must be applied to exactly balance the field's force and allow the charge to be moved.

$$dW = -\mathbf{F} \cdot d\mathbf{l} = \nabla W \cdot d\mathbf{l}$$
$$\mathbf{F} = -\nabla W \qquad \text{(where } \mathbf{F} \text{ is the field force)}$$

$\mathbf{F} = -\nabla W$ where ∇W is the gradient of the potential field W. W is a scalar function of position (a scalar field) and ∇W is a vector field. ∇ is the del operator, kind of a vector derivative operator. It has different forms in the coordinate systems we use.

In rectangular coordinates, $\nabla = \mathbf{a}_x \frac{\partial}{\partial x} + \mathbf{a}_y \frac{\partial}{\partial y} + \mathbf{a}_z \frac{\partial}{\partial z}$

In cylindrical coordinates, $\nabla = \mathbf{a}_\rho \frac{\partial}{\partial \rho} + \mathbf{a}_\phi \frac{1}{\rho}\frac{\partial}{\partial \phi} + \mathbf{a}_z \frac{\partial}{\partial z}$

In spherical coordinates, $\nabla = \mathbf{a}_r \frac{\partial}{\partial r} + \mathbf{a}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \mathbf{a}_\phi \frac{1}{r \sin\theta}\frac{\partial}{\partial \phi}$

$$\mathbf{F} = -\nabla W = -\left(\mathbf{a}_x \frac{\partial W}{\partial x} + \mathbf{a}_y \frac{\partial W}{\partial y} + \mathbf{a}_z \frac{\partial W}{\partial z} \right)$$

Dividing the electrostatic energy by charge gives the potential (work per unit charge). Taking the gradient would then result in the force per unit charge, or electric field.

$$\frac{\mathbf{F}}{q} = -\nabla\frac{W}{q} \quad \Rightarrow \quad \mathbf{E} = -\nabla V = -\left(\mathbf{a}_x\frac{\partial V}{\partial x} + \mathbf{a}_y\frac{\partial V}{\partial y} + \mathbf{a}_z\frac{\partial V}{\partial z}\right)$$

Restating, an alternate and equivalent way of thinking about the electric field is that it is the negative gradient of the electrostatic potential, which is the electrostatic potential energy per unit charge.

$$\mathbf{E} = \frac{-\nabla W}{q} = -\nabla\frac{W}{q} = -\nabla V$$

$$\mathbf{E} = \frac{-\nabla W}{q} = \frac{\mathbf{F}}{q}$$

Electrostatic potential is the electrostatic potential energy divided by charge. The SI unit for energy per unit charge is volts, V = J/C. Since the electric field is the negative gradient of the electrostatic potential (also referred to as just the "potential"), equivalent units for the electric field are volts per meter (the del operator has units of m^{-1}). The electric field can therefore be specified with two equivalent sets of units, J/C or V/m.

Two ways in which to think of the static electric field

1. The electric field is the force per unit charge. From the example above:

$$\mathbf{E} = \frac{\mathbf{F}}{q} = \frac{q'}{4\pi\varepsilon\,|\mathbf{r} - \mathbf{r}'|^2}\frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|}$$

   For q' at the origin ($\mathbf{r}' = 0$):

$$\mathbf{E} = \frac{q'}{4\pi\varepsilon\, r^2}\mathbf{a}_r \qquad \text{(SI units of } \mathbf{E} \text{ are N/C)}$$

2. The electric field is potential difference per unit length. From the example above:

$$\mathbf{E} = -\nabla V = -\nabla\left(\frac{q'}{4\pi\varepsilon\, r}\right) \qquad \text{(SI units of } \mathbf{E} \text{ are also V/m)}$$

Coulomb's law and superposition: an example

If the charge distribution is known, the superposition integral can be simplified by first finding the electrostatic potenital and then finding the electrostatic field by talking the negative gradient of the potential function.

$$V(\mathbf{r}) = \int_{q'} \frac{dq'}{4\pi\varepsilon R}$$

As was true when with using superposition integral with Coulomb's law, the integration will be a single integral if the charge is distributed along a path, will be a double integral if the charge is distributed on a surface, will be a triple integral if the charge is distributed in a volume, and will be a sum if the distribution is a collection of discrete charges.

Example—E in spherical and rectangular coordinates

Consider a point charge q' at the origin. Use $\mathbf{E} = -\nabla V$ to find $\mathbf{E}$ given $V = \frac{q'}{4\pi\varepsilon\, r}$. This will be done first using rectangular coordinates and then using spherical coordinates.

i) Using rectangular coordinates, $V = \frac{q'}{4\pi\varepsilon\, r} = \frac{q'}{4\pi\varepsilon\left(x^2 + y^2 + z^2\right)^{1/2}}$

$$\mathbf{E} = -\nabla V = -\left(\mathbf{a}_x \frac{\partial V}{\partial x} + \mathbf{a}_y \frac{\partial V}{\partial y} + \mathbf{a}_z \frac{\partial V}{\partial z}\right)$$

$$= -\frac{q'}{4\pi\varepsilon}\left[\mathbf{a}_x \frac{\partial}{\partial x}\left(x^2 + y^2 + z^2\right)^{-1/2} + \mathbf{a}_y \frac{\partial}{\partial y}\left(x^2 + y^2 + z^2\right)^{-1/2} + \mathbf{a}_z \frac{\partial}{\partial z}\left(x^2 + y^2 + z^2\right)^{-1/2}\right]$$

$$= \frac{q'}{4\pi\varepsilon}\left[\mathbf{a}_x \frac{x}{\left(x^2 + y^2 + z^2\right)^{3/2}} + \mathbf{a}_y \frac{y}{\left(x^2 + y^2 + z^2\right)^{3/2}} + \mathbf{a}_z \frac{z}{\left(x^2 + y^2 + z^2\right)^{3/2}}\right]$$

ii) Using spherical coordinates, $\nabla V = \frac{\partial V}{\partial r}\mathbf{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\mathbf{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi$

$$\mathbf{E} = -\frac{q'}{4\pi\varepsilon}\left[\mathbf{a}_r \frac{\partial}{\partial r}\left(\frac{1}{r}\right) + \mathbf{a}_\theta \frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{1}{r}\right) + \mathbf{a}_\varphi \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\left(\frac{1}{r}\right)\right] = -\frac{q'}{4\pi\varepsilon}\mathbf{a}_r\left(-\frac{1}{r^2}\right) = \frac{q'}{4\pi\varepsilon r^2}\mathbf{a}_r$$

The electric field derived is the same regardless of the CS 0x9 <</M25 d[7 26474<0n/M25 d[7 26474<0n/M20
q'00B5 T 67 T99-Tf 9.714

Potential Surfaces ☼

If the directional derivative of the electric field is zero for a particular direction, this indicates that the potential does not change in that direction. Since the electric field points in some direction, any direction that is perpendicular to the electric field give a directional derivative of zero. Taken overall all space these form a surface, an equipotential surface, which is perpendicular to the electric field (or, which is the same thing, to the negative gradient of the potential).

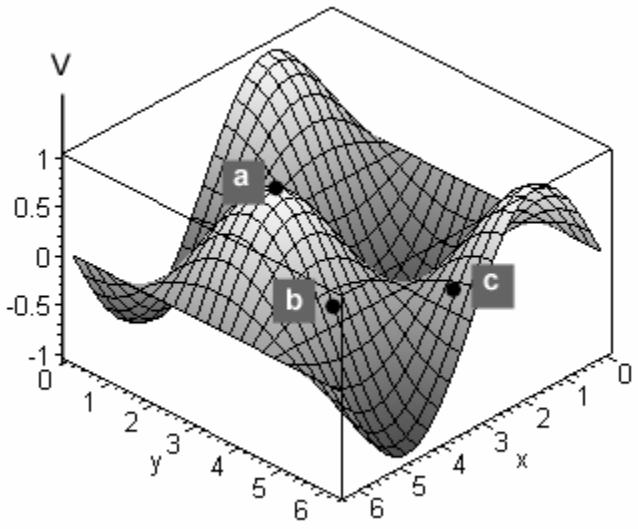
The equipotential surface is perpendicular to the gradient (or the negative of the gradient) so that the electric field is always perpendicular to equipotential surfaces

The gradient of the potential (the negative of the electric field) is perpendicular to these equipotential surfaces which are analogous to contour maps which show lines of constant elevations. In the case of contour maps, the gradient of the gravitational potential would be perpendicular to these elevation contours.

Using the physical definition of potential and the gradient operation, consider the diagram below which shows a potential, V which depends only on x and y. *This plot provides information regarding energy, electric field and the charge distribution required to produce it.*

Graphically characterize the potential gradient and the field at points a, b, and c.

Sketch some representative equipotential contours.

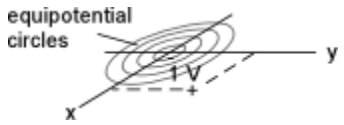


Example—energy, potential, and polarity

Given the potential $V(\mathbf{r}) = 0.5\ x^2 + 0.5\ y^2$ volts, find the energy required to move 1 coulomb of charge from (000) to (110) meters.

At (000), the charge's potential energy is 0 J. At (110), its potential energy is 1 J. Therefore, it takes 1 J to move the charge from (000) to (110) meter.

What is the potential difference between (000) and (110)? It is 1J / 1C = 1V. What is its polarity? Where are the positive and negative signs? This is every bit as important is the magnitude. If the voltage difference is 1 V, where are the positive and negative signs?



As the +1C of charge is being pushed by an external force from (000) to (110), work is done on the charge and is transformed into the charge's potential energy.

Potential energy and electrostatic potential

One might ask whether the energy required in moving the charge was independent of the path taken. Is the energy required to move the charge along $\mathbf{a}_x$ from (000) to (100) and then along $\mathbf{a}_y$ to (110) the same as that required to move the charge (000) in a straight line to (110)?

To demonstrate whether or not it is—and it is in this case—one must compare the **line integrals** along the two paths. The integrand in this case is $dW = -\mathbf{F} \cdot d\mathbf{l} = -q\mathbf{E} \cdot d\mathbf{l}$

$$W_{required} = \int_{start}^{end} dW = \int_{start}^{end} -q\mathbf{E} \cdot d\mathbf{l}$$

Line integrals, potential energy and voltage drops

The differential of energy required to move the charge is a given direction is $dW_i = -q\mathbf{E} \cdot d\mathbf{l}_i$. Integration is used to find the total energy required to move along a given path. Since the differential may be a function of position, the integration must be performed in a manner which incorporates the dependence. The result is a line integral in which the effect of the path on the integrand is taken into account.

In terms of potential, the **potential rise** in a given direction is simply the energy divided by the charge, $dV_i = -\mathbf{E} \cdot d\mathbf{l}_i$. A path integral would then be used to determine the total voltage rise along the path.

$$V_{ab} = -\int_b^a \mathbf{E} \cdot d\mathbf{l}, \text{ which is numerically approximated as } V_{ab} \cong -\sum_{i=1}^{N} \mathbf{E} \cdot \mathbf{a}_i \Delta \mathbf{l}$$

Line Integrals

Line Integrals are evaluated using these three steps

Step 1 Form the dot product of the integrand and the differential length such as $dW_i = -q\mathbf{E} \cdot d\mathbf{l}_i$ or $dV_i = -\mathbf{E} \cdot d\mathbf{l}_i$.

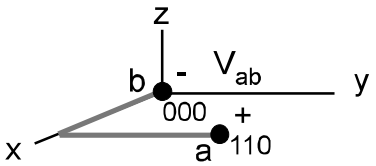
Step 2 Incorporate the effects of the path on the integrand and on the path

- Are any of the variables constant over the path? If so this would allow them to be treated as constants and their differentials would be zero.
- Are any variables zero over the path?

Step 3 Integrate

Example—line integral: rectangular coordinates, non-conservative field

Given the electric field below, find V_{ab} between a = (110) and b = (000) along the path (000) $\overrightarrow{\mathbf{a}_x}$ (100) $\overrightarrow{\mathbf{a}_y}$ (110).



$$\mathbf{E} = x\,\mathbf{a}_x + 2z\,\mathbf{a}_y - 3x\,\mathbf{a}_z \text{ V/m}$$

Step 1

Form the integrand by taking the dot, or scalar, product of –**E** and d**l**.

$$dV = -\mathbf{E} \cdot d\mathbf{l} = -\left(x\,\mathbf{a}_x + 2z\,\mathbf{a}_y - 3x\,\mathbf{a}_z\right) \cdot \left(dx\,\mathbf{a}_x + dy\,\mathbf{a}_y + dz\,\mathbf{a}_z\right)$$

$$dV = -x\,dx - 2z\,dy + 3x\,dz$$

Step 2

Incorporate the effect of the path on the integrand,

$$V_{ab} = \int_{x=0}^{1} (-x\,dx - 2z\,dy + 3x\,dz)\Big|_{\substack{y=z=0\\dy=dz=0}} + \int_{y=0}^{1} (-x\,dx - 2z\,dy + 3x\,dz)\Big|_{\substack{x=1\\z=0\\dx=dz=0}}$$

$$V_{ab} = \int_{x=0}^{1} -x\,dx + \int_{y=0}^{1} 0$$

Step 3

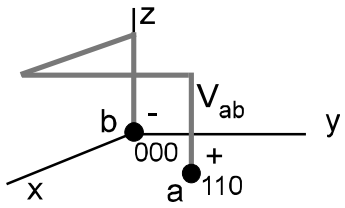
Integrate $\quad V_{ab} = \int_{x=0}^{1} -x\,dx + \int_{y=0}^{1} 0 = -\frac{x^2}{2}\Big|_0^1 = -\frac{1}{2}$ V

It would require 1C (V_{ab}) = -0.5 J to move 1 C from b to a.

Does the potential depend on the path taken?

Consider taking the path below in the same field

(000) $\vec{\mathbf{a}}_z$ (001) $\vec{\mathbf{a}}_x$ (101) $\vec{\mathbf{a}}_y$ (111) $\vec{\mathbf{a}}_z$ (110)



$$\mathbf{E} = x\,\mathbf{a}_x + 2z\,\mathbf{a}_y - 3x\,\mathbf{a}_z \text{ V/m}$$

Step 1

Form the integrand by taking the dot, or scalar, product of –**E** and d**l**.

$$dV = -\mathbf{E} \cdot d\mathbf{l} = -\left(x\,\mathbf{a}_x + 2z\,\mathbf{a}_y - 3x\,\mathbf{a}_z\right) \cdot \left(dx\,\mathbf{a}_x + dy\,\mathbf{a}_y + dz\,\mathbf{a}_z\right)$$

$$dV = -x\,dx - 2z\,dy + 3x\,dz$$

Step 2

Incorporate the effect of the path on the integrand,

$$V_{ab} = \int_{z=0}^{1} (-x\,dx - 2z\,dy + 3x\,dz)\Big|_{\substack{x=y=0\\dx=dy=0}} + \int_{x=0}^{1} (-x\,dx - 2z\,dy + 3x\,dz)\Big|_{\substack{y=0\\z=1\\dy=dz=0}}$$

$$+ \int_{y=0}^{1} (-x\,dx - 2z\,dy + 3x\,dz)\Big|_{\substack{x=z=1\\dx=dz=0}} + \int_{z=1}^{0} (-x\,dx - 2z\,dy + 3x\,dz)\Big|_{\substack{x=y=1\\dx=dy=0}}$$

$$V_{ab} = \int_{z=0}^{1} 0 + \int_{x=0}^{1} -x\,dx + \int_{y=0}^{1} -2(1)\,dy + \int_{z=1}^{0} 3(1)\,dz$$

Step 3

Integrate

$$V_{ab} = -\frac{x^2}{2}\Big|_0^1 - 2y\Big|_0^1 - 3z\Big|_1^0 = \left(-\frac{1}{2} - 2 - 3\right) \text{V} = -5\frac{1}{2} \text{ V}$$

For this field, the energy required to move charge between points depends on the path taken. That is, the potential difference is path dependent.

$W_{ab} = qV_{ab}$ = 1C(-5.5 V) = - 5.5 J

Conservative Fields

What is implied if the energy required to move charge against the field depends on the path taken? What are the causes and consequences?

Physically, there must be a changing magnetic field present. If a changing magnetic field is enclosed by the loop, **Faraday's** law states that a net electromotive force is induced in the loop.

$$\oint_{\text{loop}} \mathbf{E} \cdot d\mathbf{l} = -\frac{d\phi}{dt}$$

In circuits, Kirchoff's voltage law states the sum of voltage drops about any closed loop is zero. This is assuming a special case exists. The assumption underlying KVL is that the loop does not enclose changing magnetic fields.

Assuming there are no changing magnetic fields present, the sum of the voltages about any loop is zero. The analogous statement in electromagnetics is that the path integral of the electric field about any closed path is zero.

$$\oint_{\text{loop}} \mathbf{E} \cdot d\mathbf{l} = 0$$

If this condition holds, the field is said to be conservative. No net work is required to move the charge about any closed loop, which implies a single-valued electrostatic potential function (energy per unit charge) can be defined.

If the field is derived from an electrostatic potential, the field must be conservative.

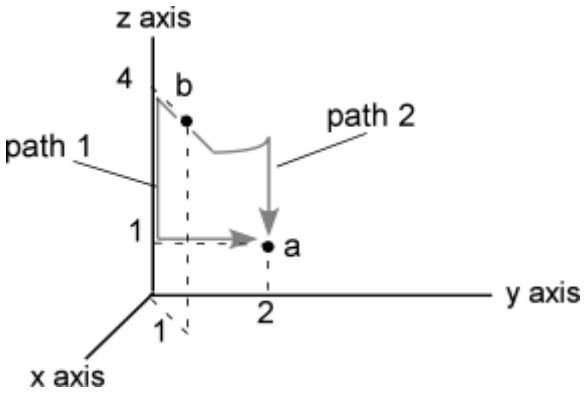
Example—cylindrical coordinates, conservative field ☼

$V(\mathbf{r}) = 0.5\, x^2 + 0.5\, y^2 = 0.5\, \rho^2$

$$\mathbf{E} = -\nabla V = -\left(\frac{\partial V}{\partial \rho}\mathbf{a}_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi + \frac{\partial V}{\partial z}\mathbf{a}_z \right)$$

$\mathbf{E} = -\rho\, \mathbf{a}_\rho$ V/m

Determine the work required to move -1 C from b = (ρ, ϕ, z) = (1, π/4, 4) to a = (2, π/2, 1) via the two paths shown.



Path 1

work required to move q from b to a is $W = qV_{ab}$

$$V_{ab} = \int_b^a -\mathbf{E} \cdot d\mathbf{l} = \int_a^b \mathbf{E} \cdot d\mathbf{l}$$

$$V_{ab} = \int_{\rho=2}^{0} -\rho\mathbf{a}_\rho \cdot \left(dz\,\mathbf{a}_z + \rho d\phi\,\mathbf{a}_\phi + d\rho\,\mathbf{a}_\rho\right) + \int_{z=1}^{4} -\rho\mathbf{a}_\rho \cdot \left(dz\,\mathbf{a}_z + \rho d\phi\,\mathbf{a}_\phi + d\rho\,\mathbf{a}_\rho\right) + \int_{\rho=0}^{1} -\rho\mathbf{a}_\rho \cdot \left(dz\,\mathbf{a}_z + \rho d\phi\,\mathbf{a}_\phi + d\rho\,\mathbf{a}_\rho\right)$$

$$V_{ab} = \int_{\rho=2}^{0} -\rho\mathbf{a}_\rho \cdot d\rho\,\mathbf{a}_\rho + \int_{z=1}^{4} -\rho\mathbf{a}_\rho \cdot dz\,\mathbf{a}_z + \int_{\rho=0}^{1} -\rho\mathbf{a}_\rho \cdot d\rho\,\mathbf{a}_\rho = -\left(\left.\frac{\rho^2}{2}\right|_2^0 + \left.\frac{\rho^2}{2}\right|_0^1\right) = (2 - 0.5)\ \text{V}$$

$$W = qV_{ab} = (-1\ \text{C})(1.5\ \text{V}) = -1.5\ \text{J}$$

Path 2

$$V_{ab} = \int_{z=1}^{4} -\rho\mathbf{a}_\rho \cdot \left(dz\,\mathbf{a}_z + \rho d\phi\,\mathbf{a}_\phi + d\rho\,\mathbf{a}_\rho\right) + \int_{\phi=\pi/2}^{\pi/4} -\rho\mathbf{a}_\rho \cdot \left(dz\,\mathbf{a}_z + \rho d\phi\,\mathbf{a}_\phi + d\rho\,\mathbf{a}_\rho\right) + \int_{\rho=2}^{1} -\rho\mathbf{a}_\rho \cdot \left(dz\,\mathbf{a}_z + \rho d\phi\,\mathbf{a}_\phi + d\rho\,\mathbf{a}_\rho\right)$$

$$V_{ab} = \int_{z=1}^{4} -\rho\mathbf{a}_\rho \cdot dz\,\mathbf{a}_z + \int_{\phi=\pi/2}^{\pi/4} -\rho\mathbf{a}_\rho \cdot \rho d\phi\,\mathbf{a}_\phi + \int_{\rho=2}^{1} -\rho\mathbf{a}_\rho \cdot d\rho\,\mathbf{a}_\rho = -\left.\frac{\rho^2}{2}\right|_{\rho=2}^{1} = (-0.5 + 2)\ \text{V}$$

$$W = qV_{ab} = (-1\ \text{C})(1.5\ \text{V}) = -1.5\ \text{J}$$

Conservative and non-conservative electric fields

What causes the potential difference to be path-dependent or path-independent? What is implied? What are the implications and consequences?

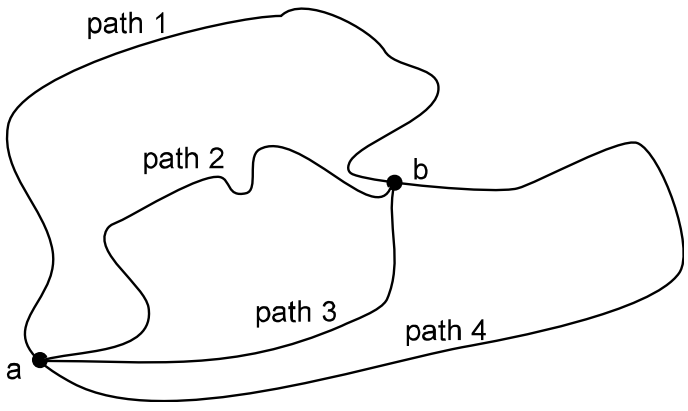
Terminology: If the potential difference between two points is independent of the path taken, the answer is the same no matter what path is taken and the electric field is said to be **conservative**. Otherwise, if the potential difference is path dependent, the electric field is said to be **non-conservative**.

Four implications of conservative fields.

1. For a conservative field, the path integral $\int_a^b \mathbf{E} \cdot d\mathbf{l}$ is independent of path taken between a and b.
2. For a conservative field, the path integral about any closed loop $\oint \mathbf{E} \cdot d\mathbf{l}$ is zero.
3. Electric fields are conservative when no changing magnetic fields are present. The integral form of Faraday's law states $\oint \mathbf{E} \cdot d\mathbf{l} = -\frac{d\phi}{dt}$
4. Any electric field derived from an electrostatic potential, $\mathbf{E} = -\nabla V$, is a conservative field.

For a conservative field, the following path integrals would be equal.

$$\int_{b,\,\text{path 1}}^{a} \mathbf{E} \cdot d\mathbf{l} = \int_{b,\,\text{path 2}}^{a} \mathbf{E} \cdot d\mathbf{l} = \int_{b,\,\text{path 3}}^{a} \mathbf{E} \cdot d\mathbf{l} = \int_{b,\,\text{path 4}}^{a} \mathbf{E} \cdot d\mathbf{l}$$



If the path integral between any two points is path independent, the path integral about any closed loop is zero.

$$\oint \mathbf{E} \cdot d\mathbf{l} = 0$$

Consider a few closed loops associated with the diagram above.

$$\underset{\substack{\rightarrow \text{ path 1} \\ \leftarrow \text{ path 2}}}{\oint} \mathbf{E} \cdot d\mathbf{l} = \underset{\text{path 1}}{\overset{a}{\underset{b}{\int}}} \mathbf{E} \cdot d\mathbf{l} + \underset{\text{path 2}}{\overset{b}{\underset{a}{\int}}} \mathbf{E} \cdot d\mathbf{l} = \underset{\text{path 1}}{\overset{a}{\underset{b}{\int}}} \mathbf{E} \cdot d\mathbf{l} - \underset{\text{path 2}}{\overset{a}{\underset{b}{\int}}} \mathbf{E} \cdot d\mathbf{l} = 0$$

$$\underset{\substack{\rightarrow \text{ path 2} \\ \leftarrow \text{ path 4}}}{\oint} \mathbf{E} \cdot d\mathbf{l} = \underset{\text{path 2}}{\overset{a}{\underset{b}{\int}}} \mathbf{E} \cdot d\mathbf{l} + \underset{\text{path 4}}{\overset{b}{\underset{a}{\int}}} \mathbf{E} \cdot d\mathbf{l} = \underset{\text{path 2}}{\overset{a}{\underset{b}{\int}}} \mathbf{E} \cdot d\mathbf{l} - \underset{\text{path 4}}{\overset{a}{\underset{b}{\int}}} \mathbf{E} \cdot d\mathbf{l} = 0$$

As stated above, the electric field is non-conservative whenever there is a changing magnetic field present in which, by Faraday's law,

$$\oint \mathbf{E} \cdot d\mathbf{l} = -\frac{d\phi}{dt}$$

where ψ_m is the magnetic flux. The magnetic flux is related to the magnetic flux density vector via a **surface integral**.

$$\phi = \underset{\text{surface}}{\iint} \mathbf{B} \cdot d\mathbf{s}$$

A consequence of having no changing magnetic fields present is that a single-valued electrostatic potential function can be defined such that $\mathbf{E} = -\nabla V$ as stated above.

Given the discussion to this point, can one say, with certainty, whether the fields below are conservative or not? *Why or why not?*

If a closed path is found for which $\oint \mathbf{E} \cdot d\mathbf{l} \neq 0$, then it can be stated with certainty that the field is non-conservative. One need look no further. One path for which $\oint \mathbf{E} \cdot d\mathbf{l} \neq 0$ is sufficient to show once and for all that the field is non-conservative.

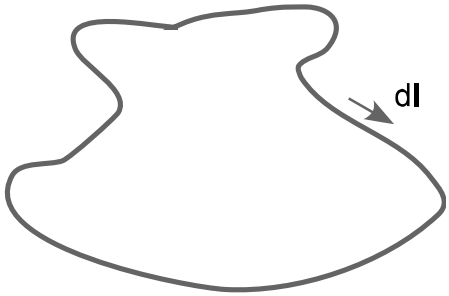
If one must use $\oint \mathbf{E} \cdot d\mathbf{l}$ to test whether the field is conservative and path after path gives $\oint \mathbf{E} \cdot d\mathbf{l} = 0$, one can never say with certainty that the field is conservative, one can only say that no path has been found yet for which $\oint \mathbf{E} \cdot d\mathbf{l} \neq 0$ and that the field may be conservative. The only certainty to be gained here would be if the electric field is given in analytic form and an electrostatic potential is found for which $\mathbf{E} = -\nabla V$. For this case one can say that the field is conservative. A better test is needed.

Conservative and non-conservative fields – curl and Stoke's theorem

The curl and Stoke's theorem allow one to say whether a field is conservative or not, once and for all.

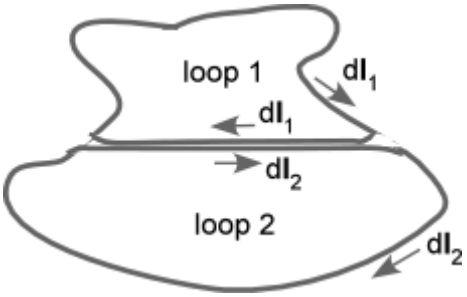
The **curl** is the third vector operation involving the del operator discussed so far (the first two were the gradient and the divergence)

Terminology: the circulation of **F** is the line integral of a vector **F** about a closed path = $\oint \mathbf{F} \cdot d\mathbf{l}$.



The term "circulation" is a term from fluid mechanics. If a fluid's velocity has a non-zero circulation, the fluid would be circulating like a whirlpool. Extending this idea beyond fluids, when any vector has a non-zero path integral, it is said to have a non-zero circulation about that path.

Notice that if the path is split into two, the sum of the two circulations is equal to the original circulation about the original loop since adjacent portions of the circulations in loop 1 and loop 2 (below) would cancel and only the outside contributions remain.



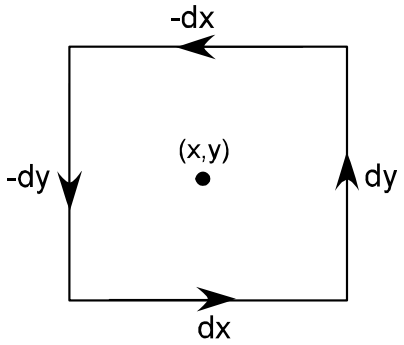
Extending this, the loop could be split into an infinite number of infinitesimal paths.

right: $F_y(x + \frac{dx}{2}, y)\, dy$

left: $F_y(x - \frac{dx}{2}, y)\, dy$

top: $F_y(x, y + \frac{dy}{2})\, dx$

bottom: $F_x(x, y - \frac{dy}{2})\, dx$



infinitesimal path

The circulation about this infinitesimal loop is

$$\sum_{\text{loop}} \mathbf{F} \cdot d\mathbf{l} = F_y(x + \frac{dx}{2}, y)\, dy - F_x(x, y + \frac{dy}{2})\, dx - F_y(x - \frac{dx}{2}, y)\, dy + F_x(x, y - \frac{dy}{2})\, dx$$

Rearranging the terms and expressing as a product of derivatives and differential surface areas,

$$\sum_{\text{loop}} \mathbf{F} \cdot d\mathbf{l} = \frac{F_y(x + \frac{dx}{2}, y)\, dy - F_y(x - \frac{dx}{2}, y)\, dy}{dx} dx - \frac{F_x(x, y + \frac{dy}{2})\, dx - F_x(x, y - \frac{dy}{2})\, dx}{dy} dy$$

$$\sum_{\text{loop}} \mathbf{F} \cdot d\mathbf{l} = \frac{F_y(x + \frac{dx}{2}, y) - F_y(x - \frac{dx}{2}, y)}{dx} dx\, dy - \frac{F_x(x, y + \frac{dy}{2}) - F_x(x, y - \frac{dy}{2})}{dy} dx\, dy$$

$$\sum_{\text{loop}} \mathbf{F} \cdot d\mathbf{l} = \left(\frac{dF_y}{dx} - \frac{dF_x}{dy} \right) dx\, dy = (\nabla \times \mathbf{F})_z\, dx\, dy$$

This quantity, $\nabla \times \mathbf{F}$, is the z-component of the curl of **F** in rectangular coordinates.

$$\nabla \times \mathbf{F} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) \mathbf{a}_x + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) \mathbf{a}_y + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \mathbf{a}_z$$

For a loop of arbitrary orientation, the circulation of F about a closed loop is not just the surface integral of the z-component of the curl, but simply the surface integral of $\nabla \times \mathbf{F}$.

$$\oint \mathbf{F} \cdot d\mathbf{l} = \iint_{\text{surface}} (\nabla \times \mathbf{F}) \cdot d\mathbf{s}$$

This relation is Stoke's theorem and can be seen to provide a physical meaning of the curl of **F**. Considering some infinitesimal path, the normal component of the curl of **F**, is the ratio of the circulation of **F** about the infinitesimal loop divided by the area of the loop.

Conservative fields and $\nabla \times \mathbf{E}$

The curl of **E** can be understood qualitatively as quite literally the “curliness” of the vector **E**. If the vector **E** has a nonzero curl, then the line integral of **E** about a closed path can be nonzero.

On the other hand, if $\nabla \times \mathbf{E}$ is zero, then by Stoke’s theorem, the circulation of **E** about any closed path must be zero and the field is therefore conservative.

$$\oint \mathbf{E} \cdot d\mathbf{l} = \iint_{\text{surface}} (\nabla \times \mathbf{E}) \cdot d\mathbf{s}$$

If $\nabla \times \mathbf{E} = 0$, it then follows that

$$\oint \mathbf{E} \cdot d\mathbf{l} = \iint_{\text{surface}} (\nabla \times \mathbf{E}) \cdot d\mathbf{s} = \iint_{\text{surface}} 0 \cdot d\mathbf{s}$$

$$\oint \mathbf{E} \cdot d\mathbf{l} = 0$$

If $(\nabla \times \mathbf{E}) = 0$, then

1) $\oint \mathbf{E} \cdot d\mathbf{l} = 0$

2) **E** is a conservative field

3) $\int_a^b \mathbf{E} \cdot d\mathbf{l}$ is independent of the path taken between a and b

4) no time-varying magnetic field is present

5) the electric potential is single-valued

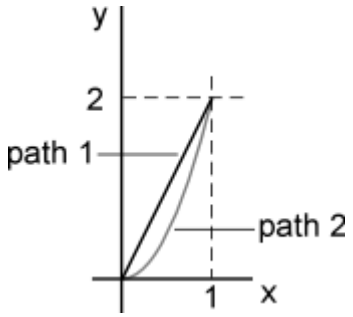
Line integrals along prescribed paths

The line integrals considered so far have always been along paths which have been split into segments where only one variable changes at a time. What is done in cases where this does not hold? What if the path is chosen in which more than one variable varies?

The answer is that "constraint equations" defining the path must be incorporated into the integrand. To illustrate, consider finding the line integral

$$\int_b^a \mathbf{E} \cdot d\mathbf{l}$$

between (x,y) = (0,0) and (1,2) for $\mathbf{E} = xy\,\mathbf{a}_x - y^2\,\mathbf{a}_y$



For rectangular coordinates, $d\mathbf{l} = dx\,\mathbf{a}_x + dy\,\mathbf{a}_y + dz\,\mathbf{a}_z \rightarrow \mathbf{E} \cdot d\mathbf{l} = xy\,dx - y^2\,dy$

Evaluation

In line integrals, there can be only one independent variable. If an integrand has more than one independent variable, constraint equations relating the variables are needed to so that a single remaining independent variable in the integrand.

Path 1: A straight line from (0,0) to (1,2). The constraint equation describing this path is $y = 2x$. The differential relation is $dy = 2\,dx$

$\mathbf{E} \cdot d\mathbf{l} = x(2x)dx - (2x)^2(2dx) = -6x^2\,dx$ **or**

$\mathbf{E} \cdot d\mathbf{l} = (\tfrac{1}{2}\,y)\,y\,(\tfrac{1}{2}dy) - y^2\,dy = -\tfrac{3}{4}\,y^2\,dy$

$$\int \mathbf{E} \cdot d\mathbf{l} = \int_0^1 -6x^2\,dx = -2 \qquad \textbf{or}$$

$$\int \mathbf{E} \cdot d\mathbf{l} = \int_0^2 -\frac{3}{4}y^2\,dy = -2$$

Path 2: The parabola $y = 2x^2$ from (0,0) to (1,2). The constraint equation is the equation of the parabola. The differential relation is $dy = 4x\,dx$

$\mathbf{E} \cdot d\mathbf{l} = x(2x^2)dx - (2x^2)^2(4x\,dx) = (2x^3 - 16x^5)d$ **or**

$\mathbf{E} \cdot d\mathbf{l} = (\tfrac{1}{2}y)^{1/2}y\,[\tfrac{1}{4}dy/(\tfrac{1}{2}y)^{1/2}] - y^2dy = (\tfrac{1}{4}y - y^2)\,dy$

$$\int \mathbf{E} \cdot d\mathbf{l} = \int_0^1 2x^3 - 16\,x^5\,dx = -2\frac{1}{6} \qquad \textbf{or}$$

$$\int \mathbf{E} \cdot d\mathbf{l} = \int_0^2 \frac{1}{4}y - y^2\,dy = -2\frac{1}{6}$$

Line integrals in 3-space

For a path integral in 3-dimensional space, two constraint equations would be required to define the path. Each of the constraint equations in three space would define a surface and their intersection (obtained through their simultaneous solution) would define the path of integration.

The integration would be with respect to whichever of the three variables is not eliminated with the two constraint equations.

In fact, we could consider the path integrals above as being performed in 3-space, where one of the constraint equations is $z = 0$.

Poisson's and Laplace's equations

The electrostatic field is the negative gradient of the electrostatic potential,

$$\mathbf{E} = -\nabla V$$

For a region bounded by surfaces of known potential, casting the problem in the form of a boundary value problem using potential can be the most natural solution path. Starting with Gauss' law, the use of $\mathbf{E} = -\nabla V$ gives the differential equation in V known as Poisson's equation.

Solving the boundary value problem for the potential avoids having to solve the vector differential equation directly. It offers an "end run" by first solving for the potential, the negative gradient of which is electric field.

$$\nabla \cdot (-\nabla V) = \frac{\rho_v}{\varepsilon} \qquad \left(\text{using } \mathbf{E} = -\nabla V \text{ in the point form of Gauss' law } \nabla \cdot \mathbf{E} = \frac{\rho_v}{\varepsilon}\right)$$

$$\nabla \cdot \nabla V = -\frac{\rho_v}{\varepsilon}$$

$\nabla \cdot \nabla V$ is the Laplacian operator operating on V and is written, $\nabla \cdot \nabla V = \nabla^2 V$.

The result is **Poisson's equation**.

$$\nabla^2 V = -\frac{\rho_v}{\varepsilon}$$

In rectangular coordinates, this implies,

$$\nabla \cdot \nabla V = \nabla \cdot \left(\mathbf{a}_x \frac{\partial}{\partial x} + \mathbf{a}_y \frac{\partial}{\partial y} + \mathbf{a}_z \frac{\partial}{\partial z}\right) \cdot \left(\mathbf{a}_x \frac{\partial V}{\partial x} + \mathbf{a}_y \frac{\partial V}{\partial y} + \mathbf{a}_z \frac{\partial V}{\partial z}\right) = -\frac{\rho_v}{\varepsilon}$$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -\frac{\rho_v}{\varepsilon}$$

For $\rho_v = 0$, the result is **Laplace's equation**.

$$\nabla^2 V = 0$$

Poisson's and Laplace's equation and the subsequent gradient operation give the electric field only for electrostatics—when the source of the fields, charge, is at rest.

When time-varying currents are present, Faraday's law shows that the line integral of the electric field about a closed path is no longer zero—the electric field is no longer a conservative one. A non-conservative field cannot be expressed as the negative gradient of a scalar potential. In this case, the concept of a vector potential is often introduced. Here, the static case is treated.

Laplace's Equation

Laplace's equation is present is many branches of science and engineering.

$$\nabla^2 V = 0$$

In electromagnetics, Laplace's equation gives the electrostatic potential from which the electric field can be found ($\mathbf{E} = -\nabla V$) given no changing magnetic fields are present.

In rectangular coordinates Laplace's equation reads,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) V = 0$$

Laplace's Equation will be solved analytically for the case of variations with involving one variable. (When more than one variable varies, the techniques of partial differential equations must be used.)

Consider a potential for which the potential is a function of z and which is independent of x and y.

$$\frac{\partial^2 V}{\partial x^2} = \frac{d^2 V}{dy^2} = 0$$

solving,

$$\frac{d^2 V}{dz^2} = \frac{d}{dz}\frac{dV}{dz} = 0 \quad \rightarrow \quad \int d\frac{dV}{dz} = \int 0 dz = A$$

$$\frac{dV}{dz} = A \quad \rightarrow \quad \int dV = \int A\, dz$$

$$V(z) = Az + B$$

applying the bc's of V(0) = 0 and V(L) = V_s

$$V(0) = B = 0$$

$$V(L) = V_s = AL \quad \rightarrow \quad A = \frac{V_s}{L}$$

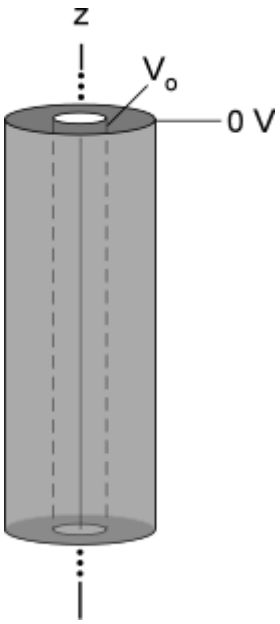
$$V(z) = \frac{V_s\, z}{L}$$

Using the gradient

$$\mathbf{E} = -\frac{\partial V}{\partial z}\mathbf{a}_z = -\frac{V_s}{L}\mathbf{a}_z$$

Laplace's equation in cylindrical coordinates

Consider two concentric cylinders that are PECs (inner radius a, outer radius b). Let the voltage of the outer cylinder be 0 and the voltage of the inner cylinder be V_o.



$$V(b) = 0$$
$$V(a) = V_o$$

Laplace's Equation in cylindrical coordinates

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial\phi^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

For this situation, there are no variations in θ or ϕ, and Laplace's Equation reduces to

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V}{\partial\rho}\right) = \frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dV}{d\rho}\right) = 0$$

Solving this

$$\int d\left(\rho\frac{dV}{d\rho}\right) = \int 0\,\rho\,d\rho \qquad \rightarrow \qquad \rho\frac{dV}{d\rho} = A$$

$$\int dV = \int\frac{A}{\rho}d\rho \qquad \rightarrow \qquad V = A\ln(\rho) + B$$

Applying the boundary conditions (bc's)

$$V(b) = A\ln(b) + B = 0 \qquad \rightarrow \qquad B = -A\ln(b)$$

$$V(a) = A\ln(a) - A\ln(b) = V_o \qquad \rightarrow \qquad A = -\frac{V_o}{\ln\left(\frac{a}{b}\right)}$$

$$A = \frac{V_o}{\ln\left(\frac{b}{a}\right)} \text{ and } B = -\frac{V_o}{\ln\left(\frac{b}{a}\right)}\ln(b)$$

$$V = \frac{V_o}{\ln\left(\frac{b}{a}\right)}\ln(\rho) - \frac{V_o}{\ln\left(\frac{b}{a}\right)}\ln(b) = V_o\frac{\ln\left(b/\rho\right)}{\ln\left(b/a\right)}$$

Finding **E**

$$\mathbf{E} = -\nabla V$$

$$\nabla V = \frac{\partial V}{\partial \rho}\mathbf{a}_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi + \frac{\partial V}{\partial z}\mathbf{a}_z$$

$$\mathbf{E} = \frac{\partial V}{\partial \rho}\mathbf{a}_\rho = \frac{\partial}{\partial \rho}\left[\frac{V_o}{\ln\left(\frac{b}{a}\right)}\ln(\rho) - \frac{V_o}{\ln\left(\frac{b}{a}\right)}\ln(b)\right]\mathbf{a}_\rho$$

$$\mathbf{E} = \frac{V_o}{\ln\left(b/a\right)}\frac{1}{\rho}\mathbf{a}_\rho$$

Numeric Techniques in solving Laplace's Equation

In rectangular coordinates, Laplace's equation reads

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

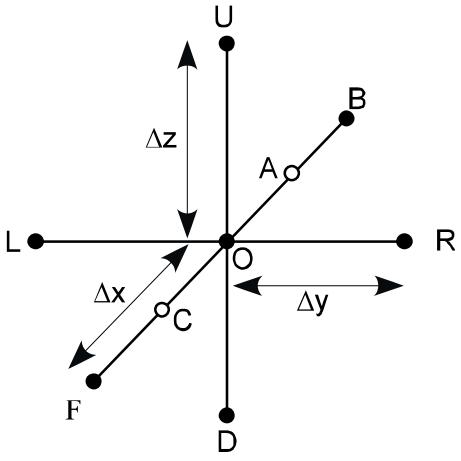
Defining potentials at points on a grid.

Evaluating the first derivative with respect to x at A

$$\left.\frac{\partial V}{\partial x}\right|_A \simeq \frac{V_O - V_B}{\Delta x}$$

and **at C**

$$\left.\frac{\partial V}{\partial x}\right|_C \simeq \frac{V_F - V_O}{\Delta x}$$



Evaluate the **2nd derivative with respect to x at O**.

$$\left.\frac{\partial^2 V}{\partial x^2}\right|_O \simeq \frac{\left.\frac{\partial V}{\partial x}\right|_C - \left.\frac{\partial V}{\partial x}\right|_A}{\Delta x}$$

substituting

$$\left.\frac{\partial^2 V}{\partial x^2}\right|_O \simeq \frac{V_F + V_B - 2V_O}{(\Delta x)^2}$$

Similar expressions can be found for the second derivatives with respect to y and z.

$$\left.\frac{\partial^2 V}{\partial y^2}\right|_O \simeq \frac{V_R + V_L - 2V_O}{(\Delta y)^2}$$

$$\left.\frac{\partial^2 V}{\partial z^2}\right|_O \simeq \frac{V_U + V_D - 2V_O}{(\Delta z)^2}$$

Take $\Delta x = \Delta y = \Delta z = \Delta$.

The numerical approximation for Laplace's equation reads.

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

$$\nabla^2 V \simeq \frac{V_F + V_B + V_R + V_L + V_U + V_D - 6V_O}{\Delta^2} = 0$$

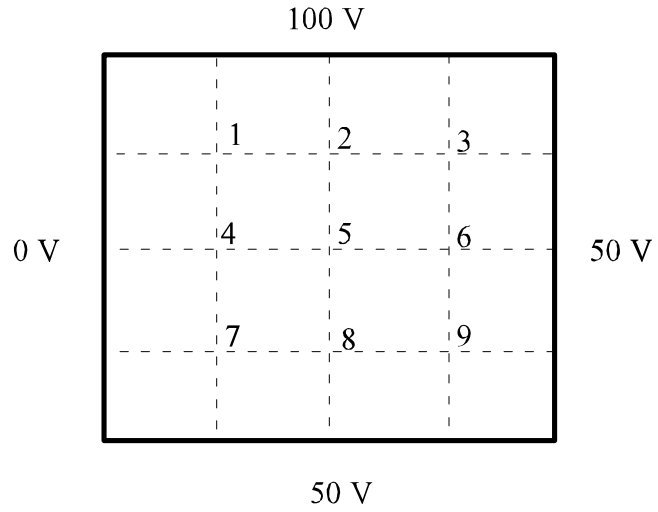
This gives the reasonable result that the voltage at O is just the average of the surrounding voltages. Why is this reasonable?

$$\frac{V_F + V_B + V_R + V_L + V_U + V_D}{6} \simeq V_o$$

For the 2D case

$$\frac{V_R + V_L + V_U + V_D}{4} \simeq V_o$$

Linear equations



Develop nine equations and nine unknowns (V_1 through V_9)

1) $V_R + V_L + V_U + V_D = 4V_o \quad \Rightarrow \quad V_2 + 0\ V + 100\ V + V_4 = 4V_1$

2)

3)

4)

5)

6)

7)

8)

9)

Once the potential voltage at the nodes is know, linear interpolation can be used to find the potential at any point.

The general guideline that can be relied upon in homogenous resistors is that the potential at any point α is the average of the potential at surrounding points arrayed symmetrically about the point α.

Using MS Excel

For 2D problems, iteration can be implemented on a spreadsheet. For example, in the problem above, twenty-one (21) cells would be required… 9 for V_1 through V_9 and 12 for the boundaries. The potential of each of the cells corresponding to V_1 through V_9 is calculated as the average of the four surrounding cells.

To begin, give the boundary cells their voltages and the interior cells zero volts. Begin iterating. V_1 through V_9 will usually stabilize quickly.

Table as entered

	100	100	100	
0	0	0	0	50
0	0	0	0	50
0	0	0	0	50
	50	50	50	

After 1st iteration (circular reference enabled by clicking iteration under tools-options-calculation)

	100	100	100	
0	25	31.3	45.3	50
0	6.25	9.38	26.2	50
0	14.1	18.4	36.1	50
	50	50	50	

After 100th iteration

	100	100	100	
0	50	67	67.9	50
0	33	50	54.5	50
0	32.1	45.5	50	50
	50	50	50	

Ohm's law and resistance

The ratio of current density to electric field in a material defines conductivity, σ.

$$\mathbf{J} = \sigma\mathbf{E}$$

This relation is the **point form of Ohm's law**. It relates current density (A/m^2) to the electric field present (V/m) just as Ohm's law relates current (A) to voltage difference (V).

$$V = IR \qquad \rightarrow \qquad I = \frac{1}{R}V$$

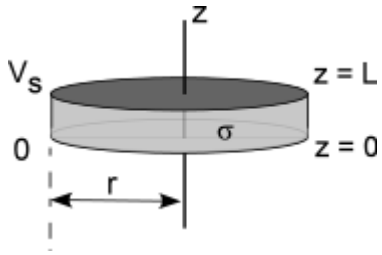
The units of conductivity are

$$[\sigma] = \frac{1}{\Omega m} = \frac{S}{m}$$

where S is Siemens, the SI unit for conductance and equal to Ω^{-1}.

Example—resistor

Find the resistance of the resistor shown.



1) Solving Laplace's equation (assuming no dependence on x or y)

$$V(z) = \frac{V_s}{L}z$$

2) The electric field is the negative gradient of the potential.

$$\mathbf{E} = -\nabla V = -\mathbf{a}_z \frac{\partial}{\partial z}\left(\frac{V_s}{L}z\right) = -\frac{V_s}{L}\mathbf{a}_z$$

3) The current density is immediately known $\mathbf{J} = \sigma\mathbf{E}$.

4) The current density can be integrated to find the total current in the resistor.\

$$I = \iint_{surface} \mathbf{J} \cdot d\mathbf{s}$$

where d**s** is in the direction of the desired current (here $-\mathbf{a}_z$)

$$I = \int_{\rho=0}^{r}\int_{\phi=0}^{2\pi}\left(-\sigma\frac{V_s}{L}\,\mathbf{a}_z\right) \cdot -\rho\, d\rho\, d\phi\, \mathbf{a}_z = \sigma\frac{\pi r^2}{L}V_s$$

5) At this point, the total current, I, is known in terms of the potential difference applied across the resistor, V_s. The ration V_s/I is the resistor's resistance.

$$R = \frac{V_s}{I} = \frac{L}{\sigma\pi r^2} = \frac{L}{\sigma A}$$

where $A = \pi r^2$ is the resistor's cross-sectional area.

Steps 2 through 5 provide an outline of a general technique with which to find a resistor's resistance. Here **E** is found from the potential, but as we've done, it is also possible to determine the electric field from a charge distribution using Coulomb's law and superposition or with Gauss' law

.

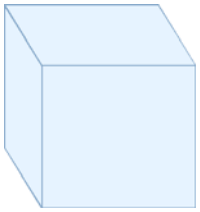
$$\mathbf{E}\Big|_{\mathbf{J}=\sigma\mathbf{E}} \rightarrow \mathbf{J}\Big|_{I=\iint \mathbf{J}\cdot d\mathbf{s}} \rightarrow I\Big|_{R=\frac{V_s}{I}} \rightarrow R$$

Joule's law

The differential work done on a charge q by an electric field **E** is

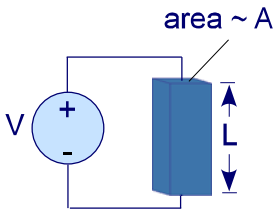
$$dW = \mathbf{F} \cdot d\mathbf{l} = q\mathbf{E} \cdot d\mathbf{l}$$

If the charge were in a vacuum, the work done on charge q would be reflected in an increase in kinetic energy of q. The charge would accelerate; the speed would increase. The mechanism of energy transfer would be between the field which, doing work, would lose energy to that of the charge which, having work done on it, would gain energy.



Consider the case where a voltage source sets up the electric field. For this case, the electric field continuously loses energy, but does not grow smaller in magnitude since the voltage source provides energy to maintain the electric field (if the electric field did not receive energy from the voltage source, its magnitude would decrease). In this case, the energy from the voltage source maintains the electric field, which does work on the moving charge and this work is transferred to the material through scattering from the material's atoms (lattice scattering). Overall, we can view this as power from the voltage source to the conducting material.

Looking at the individual charge q in an electric field **E** moving with an average velocity **v**. The time rate of work done by the electric field on the charge is



$$\frac{dW}{dt} = \mathbf{F} \cdot \frac{d\mathbf{l}}{dt} = q\mathbf{E} \cdot \mathbf{v}$$

$$P = q\mathbf{E} \cdot \mathbf{v}$$

This is the power that is transferred from the electric field **E** to thermal energy, with the charge q acting as an intermediary. Consider now the power associated with a differential charge dq.

$$dP = dq(\mathbf{E} \cdot \mathbf{v}) = \rho_v \, dv (\mathbf{E} \cdot \mathbf{v})$$

$$p = \frac{dP}{dv} = \rho_v (\mathbf{E} \cdot \mathbf{v})$$

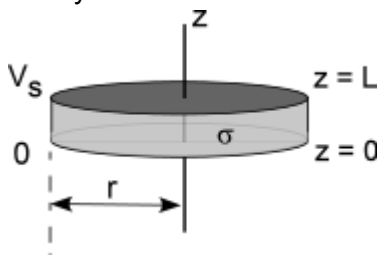
Here p is the power absorbed per unit volume, Joule's law, which is the point form of the familiar P = VI relationship giving the power absorbed by lumped circuit elements.

$$p = \mathbf{E} \cdot \rho_v \mathbf{v} = \mathbf{E} \cdot \mathbf{J}$$

$$p = \mathbf{E} \cdot \mathbf{J}$$

Example: determining resistance using Joule's law

Take the resistor that has previously been considered.



Using Ohm's law

$$P = V\,I = \frac{V^2}{R} = I^2 R \qquad \rightarrow \qquad R = \frac{V^2}{P} = \frac{P}{I^2}$$

In terms of fields

$$R = \frac{\left(\int \mathbf{E} \cdot d\mathbf{l}\right)^2}{\iiint_{volume} \sigma\, E^2 dv} = \frac{\iiint_{volume} \sigma\, E^2 dv}{\left(\iint_{surface} \sigma \mathbf{E} \cdot d\mathbf{s}\right)^2}$$

In this case, the electric field has been found to be

$$\mathbf{E} = -\nabla V = -\mathbf{a}_z \frac{\partial}{\partial z}\left(\frac{V_s}{L} z\right) = -\frac{V_s}{L}\mathbf{a}_z$$

Finding the resistance with $R = V^2/P$.

$$R = \frac{\left(\int \mathbf{E} \cdot d\mathbf{l}\right)^2}{\iiint_{volume} \sigma\, E^2 dv} = \frac{\left(\int_{z=L}^{0} -\frac{V_s}{L}\mathbf{a}_z \cdot dz\,\mathbf{a}_z\right)^2}{\int_{z=0}^{L}\int_{\varphi=0}^{2\pi}\int_{\rho=0}^{r} \sigma\left(\frac{V_s}{L}\right)^2 \rho\, d\rho\, d\varphi\, dz} = \frac{V_s^2}{\sigma\left(\frac{V_s}{L}\right)^2 L 2\pi\frac{\rho^2}{2}} = \frac{L}{\sigma\pi\rho^2} = \frac{L}{\sigma A}$$

And now with $R = P/I^2$.

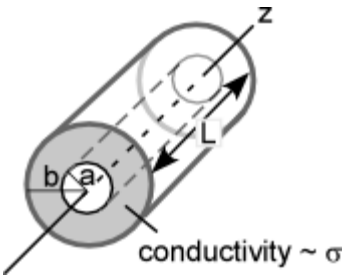
$$R = \frac{\int_{z=0}^{L}\int_{\varphi=0}^{2\pi}\int_{\rho=0}^{r} \sigma\left(\frac{V_s}{L}\right)^2 \rho\, d\rho\, d\varphi\, dz}{\left[\int_{\varphi=0}^{2\pi}\int_{\rho=0}^{r} \sigma\left(-\frac{V_s}{L}\right)\mathbf{a}_z \cdot \rho\, d\rho\, d\varphi\,(-\mathbf{a}_z)\right]^2} = \frac{\sigma\left(\frac{V_s}{L}\right)^2 L 2\pi\frac{\rho^2}{2}}{\left(\sigma\frac{V_s}{L}\right)^2\left(2\pi\frac{\rho^2}{2}\right)^2} = \frac{L}{\sigma\pi\rho^2} = \frac{L}{\sigma A}$$

Example—cylindrical resistor with Laplace

Find the resistance between two PECs forming concentric cylinders (inner radius a, outer radius b) of length L. Let the voltage of the outer cylinder be 0 and the voltage of the inner cylinder be V_o. Let the conductivity of the material between the PECs be σ. *(from symmetry, V should not vary with ϕ and the model used takes the length to be large so that variations z can be neglected)*

$V(b) = 0$

$V(a) = V_o$



Laplace's equation has been solved for cylindrical coordinates with these boundary conditions--see pages 47-48 of these notes.

$$V = \frac{V_o}{\ln\left(b/a\right)}(\ln b - \ln \rho) = V_o \frac{\ln\left(b/\rho\right)}{\ln\left(b/a\right)} = \frac{V_o}{\ln\left(b/a\right)}(\ln b - \ln \rho)$$

$$\mathbf{E} = -\nabla V = -\mathbf{a}_\rho \frac{\partial}{\partial \rho}\left[\frac{V_o}{\ln\left(b/a\right)}(\ln b - \ln \rho)\right] = \frac{V_o}{\ln\left(b/a\right)}\frac{1}{\rho}\,\mathbf{a}_\rho$$

$$R = \frac{V}{I} = \frac{\int_{+} \mathbf{E} \cdot d\mathbf{l}}{\iint_{\text{surface}} \mathbf{J} \cdot d\mathbf{s}} = \frac{\int_a^b \frac{V_o}{\ln\left(b/a\right)}\frac{1}{\rho}\,\mathbf{a}_\rho \cdot d\rho\,\mathbf{a}_\rho}{\int_{\phi=0}^{2\pi}\int_{z=z_o}^{z_o+L} \sigma \frac{V_o}{\ln\left(b/a\right)}\frac{1}{\rho}\,\mathbf{a}_\rho \cdot \rho\, dz\, d\phi\, \mathbf{a}_\rho}$$

$$R = \frac{\frac{V_o}{\ln\left(b/a\right)}\int_a^b \frac{d\rho}{\rho}}{\sigma\frac{V_o}{\ln\left(b/a\right)}\int_{\phi=0}^{2\pi}\int_{z=z_o}^{z_o+L} dz\, d\phi} = \frac{\frac{V_o}{\ln\left(b/a\right)}(\ln b - \ln a)}{\sigma\frac{V_o}{\ln\left(b/a\right)}2\pi L} = \frac{\frac{V_o}{\ln\left(b/a\right)}\ln\left(b/a\right)}{\sigma\frac{V_o}{\ln\left(b/a\right)}2\pi L}$$

$$R = \frac{\ln\left(b/a\right)}{\sigma 2\pi L}$$

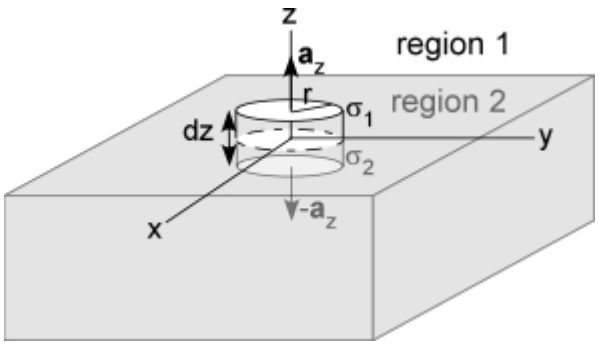
Boundary conditions ☼

Consider two materials sharing a common boundary. Material 1 has conductivity σ_1 and permittivity ε_1, and material 2 has conductivity σ_2 and permittivity ε_2. The laws of electromagnetics require certain relations for the **J**, **D**, and **E** vectors at the boundary.

Boundary conditions for **J**

$$I_{out} = \oiint_{surface} \mathbf{J} \cdot d\mathbf{s} = 0$$

$$I_{out} = \iint_{top} \mathbf{J}_1 \cdot d\mathbf{s} + \iint_{bottom} \mathbf{J}_2 \cdot d\mathbf{s} + \iint_{sides} \mathbf{J} \cdot d\mathbf{s}$$



Since the sides are infinitesimal in area, the integral over the sides is zero

$$I_{out} = \int_{\phi=0}^{2\pi} \int_{\rho=0}^{r} \mathbf{J}_1 \cdot \rho \, d\rho \, d\phi \, \mathbf{a}_z + \int_{\phi=0}^{2\pi} \int_{\rho=0}^{r} \mathbf{J}_2 \cdot \rho \, d\rho \, d\phi \, (-\mathbf{a}_z)$$

$$I_{out} = J_{1z}\left(\pi r^2\right) - J_{2z}\left(\pi r^2\right) = 0$$

$J_{1z} = J_{2z}$ or physically, $J_{1N} = J_{2N}$

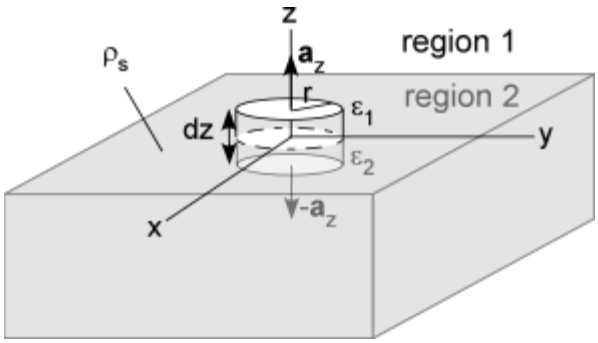
The result of conservation of magnetic flux can be expressed compactly in vector notation using the unit normal, $\mathbf{a}_n$, to the interface directed from region 2 into region 1. ($\mathbf{a}_n = \mathbf{a}_z$ above)

$$\mathbf{a}_n \cdot \left(\mathbf{J}_1 - \mathbf{J}_2\right) = 0$$

Boundary conditions for **D**

$$\psi_{out} = \oiint_{surface} \mathbf{D} \cdot d\mathbf{s} = q_{inside}$$

$$q_{inside} = \iint_{top} \mathbf{D}_1 \cdot d\mathbf{s} + \iint_{bottom} \mathbf{D}_2 \cdot d\mathbf{s} + \iint_{sides} \mathbf{D} \cdot d\mathbf{s}$$



Since the sides are infinitesimal in area, the integral over the sides is zero

$$q_{inside} = \int_{\phi=0}^{2\pi} \int_{\rho=0}^{r} \mathbf{D}_1 \cdot \rho \, d\rho \, d\phi \, \mathbf{a}_z + \int_{\phi=0}^{2\pi} \int_{\rho=0}^{r} \mathbf{D}_2 \cdot \rho \, d\rho \, d\phi \, (-\mathbf{a}_z)$$

$$\rho_s \left(\pi r^2\right) = D_{1z}\left(\pi r^2\right) - D_{2z}\left(\pi r^2\right)$$

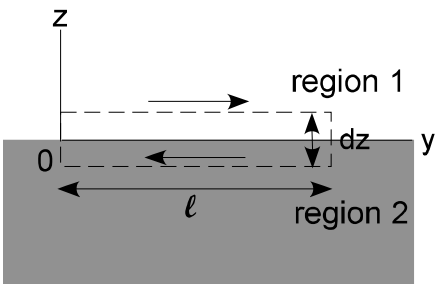
$D_{1z} - D_{2z} = \rho_s$ or physically, $D_{1N} - D_{2N} = \rho_s$

The result from Gauss' law can be expressed compactly in vector notation using the vector normal to the interface directed from region 2 into region 1. ($\mathbf{a}_n = \mathbf{a}_z$ above)

$$\mathbf{a}_n \cdot \left(\mathbf{D}_1 - \mathbf{D}_2\right) = \rho_s$$

Boundary conditions for **E**

Faraday's law states that the voltage induced about a closed loop is equal to the changing magnetic flux enclosed by the loop. Since voltage can be expressed in terms of a line integral of the electric field,



$$\oint_{\text{path}} \mathbf{E} \cdot d\mathbf{l} = -\frac{d\phi}{dt}$$

If the loop is infinitesimal, a finite amount of flux cannot be enclosed and Faraday reads,

$$\oint_{\text{path}} \mathbf{E} \cdot d\mathbf{l} = 0$$

The result from Faraday's law can be expressed compactly in vector notation using the vector normal to the interface directed from region 2 into region 1. ($\mathbf{a}_n = \mathbf{a}_z$ above)

$$\mathbf{a}_n \times (\mathbf{E}_1 - \mathbf{E}_2) = 0$$

Note that the contributions of the sides are infinitesimal due to their infinitesimal lengths.

$$\oint_{\text{path}} \mathbf{E} \cdot d\mathbf{l} = \int_{y=0}^{l} \mathbf{E}_1 \cdot dy\,\mathbf{a}_y + \int_{y=l}^{0} \mathbf{E}_2 \cdot dy\,\mathbf{a}_y = 0$$

$$E_{1y}\mathbf{l} - E_{2y}\mathbf{l} = 0 \quad \rightarrow \quad E_{1y} = E_{2y}$$

This result could be expressed more physically by noting that the a_y direction is tangential to the boundary,

$$E_{1T} = E_{2T} \quad \rightarrow \quad \mathbf{a}_n \times (\mathbf{E}_1 - \mathbf{E}_2) = 0$$

Planes and normals

A plane can be defined as all the vectors that are normal to a given vector.

$$\mathbf{N} \cdot (\mathbf{r} - \mathbf{r}_o) = (N_x\mathbf{a}_x + N_y\mathbf{a}_y + N_z\mathbf{a}_z) \cdot [(x - x_o)\mathbf{a}_x + (y - y_o)\mathbf{a}_y + (z - z_o)\mathbf{a}_z] = 0$$

$$N_x(x - x_o) + N_y(y - y_o) + N_z(z - z_o) = 0$$

$$N_x x + N_y y + N_z z = N_x x_o + N_y y_o + N_z z_o \qquad (\text{equation of plane})$$

$$\mathbf{a}_n = \mathbf{N} / N \qquad (\text{unit vector normal to plane})$$

Example

$$2x + 5y - 3z = 11 \qquad (\text{equation of plane})$$

$$\mathbf{a}_n = (2\mathbf{a}_x + 5\mathbf{a}_y - 3\mathbf{a}_z) / \sqrt{2^2 + 5^2 + 3^2} \qquad (\text{unit vector normal to plane})$$

Conductors and the perfect electrical conductor approximation

Good conductors have a large conductivity, σ. A **perfect electrical conductor** (PEC) has infinite conductivity. This implies that the electrical field inside a PEC is zero. That this must hold can be seen from the point form of Ohm's law.

$$\mathbf{J} = \sigma\mathbf{E}$$

The magnitude of **J** must remain finite, for a PEC with infinite σ, the magnitude of **E** must be zero. Of course, this condition is not exactly met within a good conductor, but it is nearly so as one can see in the example below.

Example—field required for DC currents in copper conductors

30 A is the maximum current permitted by the National Electric Code (assuming 30°C ambient temperature) for 10 AWG copper conductors. What is the electric field inside a 10 AWG conductor carrying its maximum permissible current.

radius for 10 AWG conductor = 1.29 mm

J for 30 A in 10 AWG conductor = 5.7 (10^6) A/m^2

$E = J/\sigma$ = 0.1 V/m

For a wire more on the order of that used for electronics, consider the field required for a 10 mA current in a 24 AWG copper conductor

radius for 24 AWG conductor = 0.256 mm

J for 10 mA in 24. AWG conductor = 4.86(10^4) A/m^2

$E = J/\sigma$ = 0.84 mV/m

Boundary conditions for perfect electric conductors (PECs)

1. Since E = 0 inside a PEC, E_{tan} = 0 inside a PEC. This requires that the tangential component of the electric field at the boundary of a PEC is zero. The tangential component of the electric flux density at the boundary of a PEC is zero as well.

   $E_T = 0$ and $D_T = 0$

2. Since E = 0 inside a PEC, D = 0 inside a PEC as well. This, combined with the fact that D_{tan} = 0 at the boundary of a PEC implies both the electric field and the electric flux density vector are always normal to the surface of a PEC.

   $D_N = \rho_s$ and $E_N = \rho_s/\varepsilon$

Incremental resistor ☼

Additional insight can be gained from a discussion of the incremental resistor. This can aid in numerical approaches which typically involves breaking regions into small pieces that can be approached more readily.

Looking at the relation between potential difference and electric field,

$$\int_a^b \mathbf{E} \cdot d\mathbf{l} = V_{ab}$$

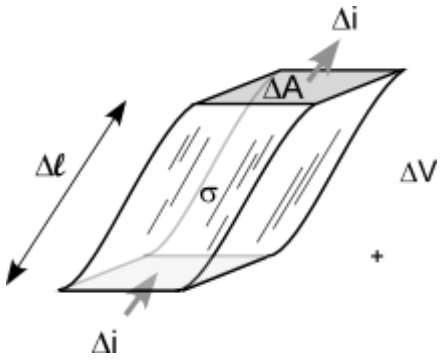
By incorporating geometry and material properties on can obtain a relationship between current and voltage for a small region of resistive material, the incremental resistor.

The sides of the incremental resistor are parallel to the current flux so that any charge entering the element also leaves the element. Also, the ends of the incremental reluctor are equipotential surfaces.

The element relationship for the incremental resistor is the ratio of the potential difference to current, Ohm's law.

$$\Delta R = \frac{\Delta V}{\Delta i} = \frac{E\,\Delta \mathbf{l}}{J\,\Delta A} = \frac{E\,\Delta \mathbf{l}}{\sigma E\,\Delta A}$$

$$\Delta R = \frac{\Delta \mathbf{l}}{\sigma\,\Delta A}$$



Using Ohm's law and the incremental resistor, it can be shown that the power relation P = VI for a resistance is consistent with the fundamental considerations used to derived Joule's law.

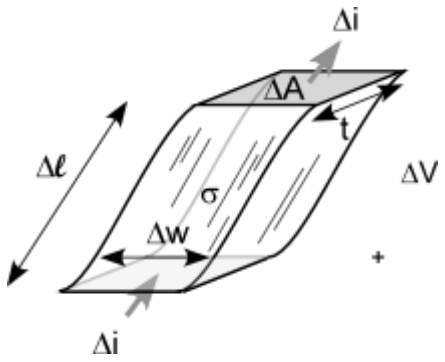
$$dP = dV\,di = dR\,di\,di = \frac{d\mathbf{l}}{\sigma\,dA}\,(J\,dA)^2 = \frac{d\mathbf{l}}{\sigma\,dA}\,(\sigma E\,dA)^2$$

$$dP = \sigma E^2\,d\mathbf{l}\,dA = \sigma E E\,dv = JE\,dv$$

$$\frac{dP}{dv} = JE = \mathbf{J} \cdot \mathbf{E} \qquad (\mathbf{J} \text{ and } \mathbf{E} \text{ are parallel})$$

Per-square resistance

Suppose the incremental resistance is two-dimensional in character with a thickness t.



In this case, $\Delta R = \dfrac{\Delta \mathbf{l}}{\sigma \Delta A} = \dfrac{\Delta \mathbf{l}}{\sigma t \Delta w}$

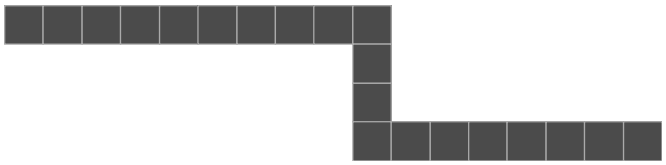
Suppose further, that $\Delta \mathbf{l} = \Delta w$, in which case

$$R_{\square} = \Delta R = \frac{1}{\sigma t}$$

Where $R_{\square}$ is referred to as resistance per square. This nomenclature is used in PCB and IC manufacturing where most structures are 2D in nature.

Example: resistance of PCB trace

Find the resistance of the PCB trace below if it is patterned from a 2-oz copper layer.



2-oz copper (2 oz copper spread over 1 ft^2) has a typical thickness of $6.81(10^{-5})$ m.

$$\rho_{cu} = 8960 \text{ kg/m}^3 = \frac{m}{v} = \frac{2 \text{ oz} \dfrac{1 \text{ lb}}{16 \text{ oz}} \dfrac{1 \text{ kg}}{2.205 \text{ lb}}}{1 \text{ ft}^2 \left(\dfrac{12 \text{ in}}{1 \text{ ft}}\right)^2 \left(\dfrac{0.0254 \text{ m}}{1 \text{ in}}\right)^2 t}$$

$$t = \frac{2 \text{ oz} \dfrac{1 \text{ lb}}{16 \text{ oz}} \dfrac{1 \text{ kg}}{2.205 \text{ lb}}}{1 \text{ ft}^2 \left(\dfrac{12 \text{ in}}{1 \text{ ft}}\right)^2 \left(\dfrac{0.0254 \text{ m}}{1 \text{ in}}\right)^2 8960 \text{ kg/m}^3} = 6.81\left(10^{-5}\right) \text{ m}$$

For $\sigma_{cu} = 5.8(10^7)$ S/m, the per-square resistance for copper is approximately 0.253 mΩ per square.

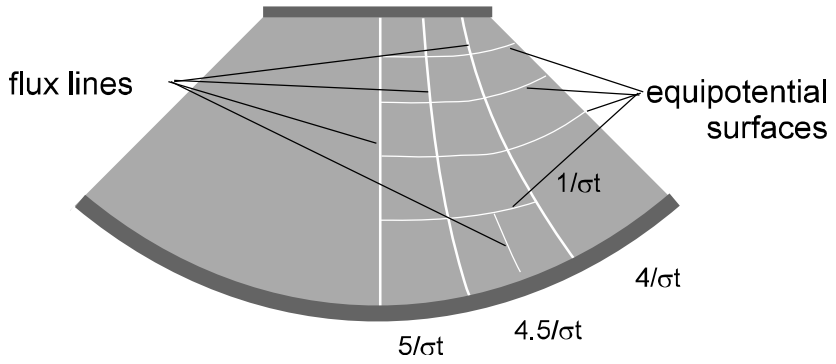
Given this per-square resistance, an estimate of the DC trace resistance can quickly be obtained for this 20-square trace.

$$R_{trace} = 20 R_{\square} = 5.06 \text{ m}\Omega$$

Resistivity estimates of irregular 2D shapes – curvilinear squares.
The approach taken above in per-square resistance can be generalized for application regarding resistors with more general cross-sections. Here shapes are chosen so that the average width is equal to the average height.

The boundaries of these “curvilinear squares” are chosen so that their sides are parallel to the current flow (so that the current "in" is equal to the current "out") and their ends are equipotential surfaces (to allow a potential difference to be defined).

Example: curvilinear squares
Assuming a thickness t, use curvilinear squares to graphically estimate the resistance of the 2D resistor shown.



Symmetry can be used to reduce the work to two identical resistance in parallel, it can be seen that the resistance of the 2D resistance is

$$R = \frac{1}{2}\,\frac{1}{\sigma t}\left(5^{-1} + 4.5^{-1} + 4^{-1}\right)^{-1} = \frac{0.744}{\sigma t}$$

This result seems reasonable since the resistor is a bit wider than it is long.

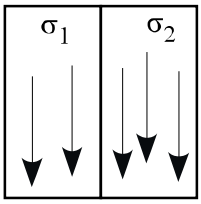
Resistance of inhomogeneous materials

What are the options when the material's conductivity varies with position?

Case one: Sides parallel to current flux lines and ends on equipotential surfaces.

If two materials are separated by surfaces parallel to current flux lines and if they share common potential differences, then the two resistances have common potentials, respective currents add, and the resistances can be treated as in parallel.

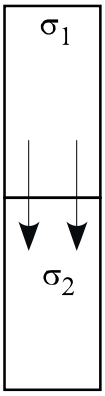
In this case, find the resistance of each resistor separately and then treat them as in parallel to find the resistance of the combination.



Case two: ends on equipotential surfaces and common currents passing through the two materials. In this case the resistances have common currents, potential differences add, and the resistances can be treated as in series.

Consider a case with two different materials.

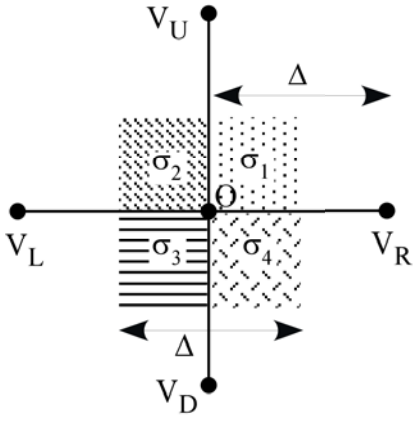
In this case, find the resistance of each resistor separately and then treat them as in series to find the resistance of the combination.



General case of non-homogeneous conductivity

Case three: Boundaries neither parallel nor perpendicular to current density flux lines.

Consider two-dimensional resistors in (xyz). The general cell is shown below.



The basic relation used is conservation of charge under the condition that one would not expect the interior of a resistive body to be capable of storing charge. That is, for a resistor, each region must satisfy $I_{out} = 0$.

Use the formula.

$$I = \mathbf{J} \cdot \text{area} = \sigma E \,(\text{area})$$

Looking at the right surface.

$$I_{out\,(right)} = \frac{V_O - V_R}{\Delta} \left\{ \left(\sigma_1 \frac{\Delta\ t}{2} \right) + \left(\sigma_4 \frac{\Delta\ t}{2} \right) \right\}$$

$$I_{out\,(right)} = \frac{V_O - V_R}{2}\ t\ (\sigma_1 + \sigma_4)$$

Performing similar calculations at the upper, the left, and the bottom surfaces, we would obtain.

$$I_{out} = \frac{V_O - V_R}{2}\ t\ (\sigma_1 + \sigma_4) + \frac{V_O - V_U}{2}\ t\ (\sigma_1 + \sigma_2) +$$
$$\frac{V_O - V_L}{2}\ t\ (\sigma_2 + \sigma_3) + \frac{V_O - V_D}{2}\ t\ (\sigma_3 + \sigma_4) = 0$$

Rearranging, one obtains,

$$2\,V_O\,(\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4) - V_R(\sigma_1 + \sigma_4) - V_U(\sigma_1 + \sigma_2) - V_L(\sigma_2 + \sigma_3) - V_D(\sigma_3 + \sigma_4) = 0$$

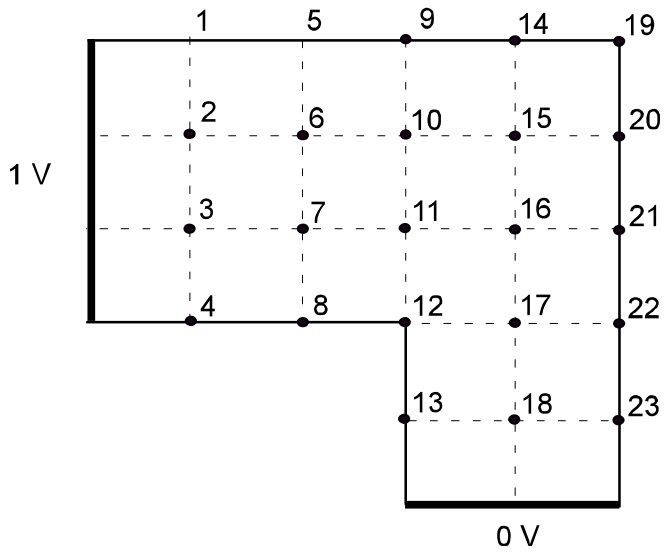
Solving for V_o,

$$V_O = \frac{V_U(\sigma_1 + \sigma_2) + V_L(\sigma_2 + \sigma_3) + V_D(\sigma_3 + \sigma_4) + V_R(\sigma_1 + \sigma_4)}{2(\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4)}$$

This is the equation that can be used to find the potentials in a two-dimensional inhomogeneous resistor. The 2D resistor is of extreme importance. Most of the resistors on the planet are 2D resistors in integrated circuits.

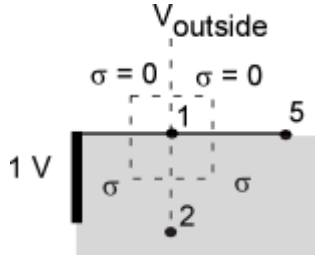
Beyond this, this technique can readily be generalized to the general inhomogeneous 3D resistor.

Consider the two dimensional resistor below.



The solution will involve writing 23 node equations. Begin with N1.

$$4V_1 = 1\text{ V} + V_5 + 2V_2$$



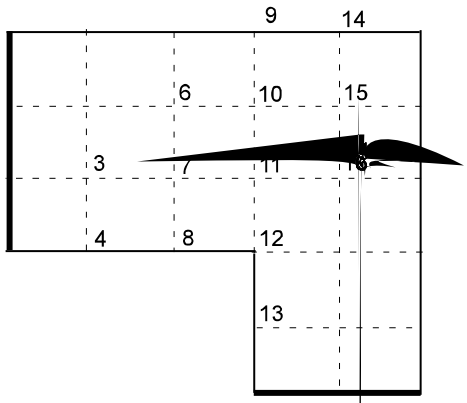
The other 22 node voltage equations,

n2: $8V_2 = 2(1V) + 2V_1 + 2V_6 + 2V_3$

n3:

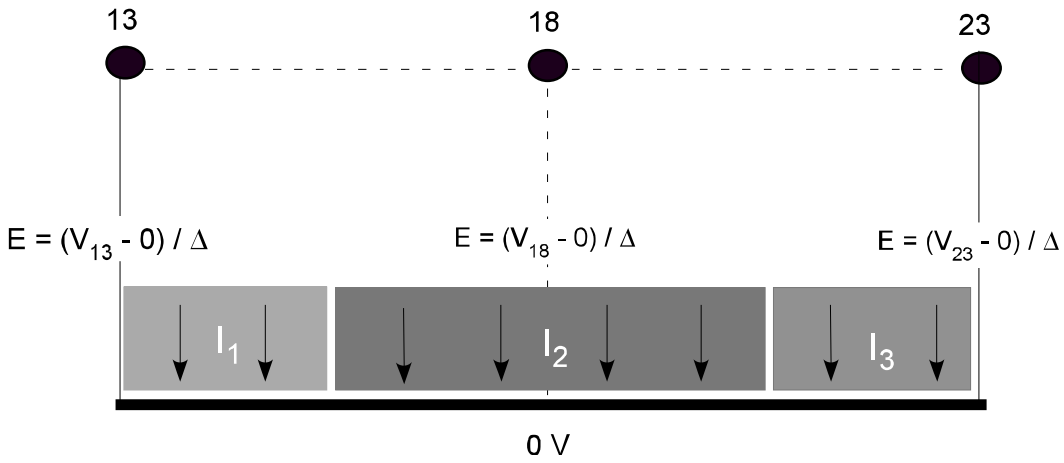
n4: $4V_2 = 1V + 2V_3 + V_8$

and so on



To find resistance, divide applied voltage by total current.

In order to do this, one must find the current density from the conductivity and the electric field. In this method a linear approximation for the electric field is found from the node potentials.



The total current can be calculated as a sum of cell currents over an area that encloses all the current. For example, to find the current that flows between N13, N18, and N23 and the 0 V electrode.

$$I = I_1 + I_2 + I_3 = J_1 A_1 + J_2 A_2 + J_3 A_3 = \sigma_1 E_1 A_1 + \sigma_2 E_2 A_2 + \sigma_3 E_3 A_3$$

$$I = \sigma \left(\frac{V_{13} - 0}{\Delta}\right) t \frac{\Delta}{2} + \sigma \left(\frac{V_{18} - 0}{\Delta}\right) t\, \Delta + \sigma \left(\frac{V_{23} - 0}{\Delta}\right) t \frac{\Delta}{2}$$

Note: Cells 13 and 23 have half their widths in material with zero conductivity.

For V_{13} = 0.218 V, V_{18} = 0.198 V, and V_{23} = 0.189 V

$$I = 0.4015\, \sigma t \text{ so that } R = 1\ V / I = 2.49/\sigma t$$

The total current could also be found at the 1V electrode.

$$I = I_1 + I_2 + I_3 + I_4$$

$$I = \sigma t\frac{1 - V_1}{2} + \sigma t\left(1 - V_2\right) + \sigma t\left(1 - V_3\right) + \sigma t\frac{1 - V_4}{2}$$

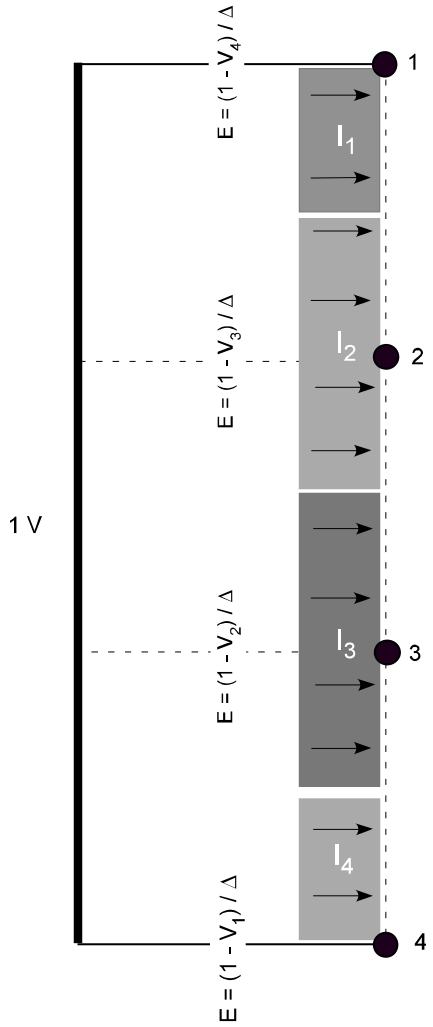
$V_1 = 0.875$ V

$V_2 = 0.871$ V

$V_3 = 0.862$ V

$V_4 = 0.855$ V

$I = 0.402\sigma t$, so that $R = 2.49/\sigma t$.



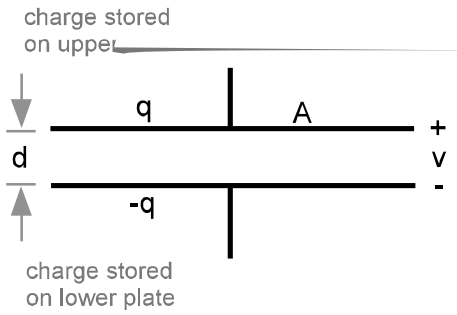
Permittivity

The ultimate source of the electric field, **E**, is charge. In free space, the ratio of **D** to **E** is $\varepsilon_o = 8.854(10^{-12})$ F/m. In matter, their ratio is affected by the ease by which the material's bound charges are polarized. In matter, bound charge can polarize which produces a component of the **D** due to polarization.

$$\mathbf{D} = \varepsilon\mathbf{E} = \varepsilon_r\varepsilon_o\mathbf{E} = \varepsilon_o\mathbf{E} + (\varepsilon_r - 1)\varepsilon_o\mathbf{E}$$

$$\mathbf{D} = \varepsilon_o\mathbf{E} + \mathbf{P} \quad \rightarrow \quad \mathbf{E} = \frac{\mathbf{D}}{\varepsilon_o} - \frac{\mathbf{P}}{\varepsilon_o}$$

In a material with bound charge, permittivity is a measure of how easily bound charge is separated – whether by the orientation of dipoles, by distortion of permanent dipoles, or by displacement of electrons – by an external electric field. The permittivity of a material is a measure of the ease in which the bound charge can be polarized, of how easily bound charge is separated under the influence of an external field. *The easier the charge is separated, the larger* ε_r.



Ionic Polarization

The electronic bonding in many materials has an ionic component due to differences in electron affinities of the material's constituents. For example, when sodium (Na) and chlorine (Cl) form sodium chloride (NaCl) or common table salt, the chloride ion attracts the bond electrons more strongly than do the sodium ions, the result being that NaCl is an ionic solid with the Na ion being positively charged and the chloride ion being negatively charged, $Na^{+}Cl^{-}$. When ionic solids are subjected to a rapidly time-varying electric field, these two ions are displaced in opposite directions due to the Coulomb force. The mass associated with this process includes some relatively small portion of the ions' mass, which is much higher than that associated with electronic polarization.

Orientational Polarization

This mechanism is associated with the movement and rotation of permanent dipoles, the outstanding example of which is the water molecule. Orientation polarization in this case refers to the movement and rotation of the dipole, not just distortion as is the case in ionic polarization. The result is that a much greater mass is typically associates with orientational polarization than is with either electronic or with ionic polarization.

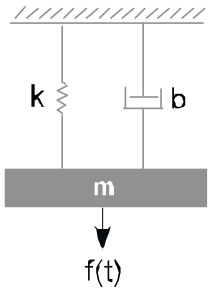
Variation of polarization with frequency

The three polarization mechanisms involve different characteristic inertias and so have different time constants. The result is that all materials have a frequency dependent permittivity It is important to know at what frequencies these polarization mechanisms begin to vary.

Think of bound charge in terms of a mass-spring-damper system. The spring constant represents the coulomb force, the mass represents the mass associated with the bound charge, and the damper represents loss mechanisms.

time domain eom: $m\ddot{x} + b\dot{x} + kx = f(t)$

s-domain eom: $X(ms^2 + bs + k) = F$

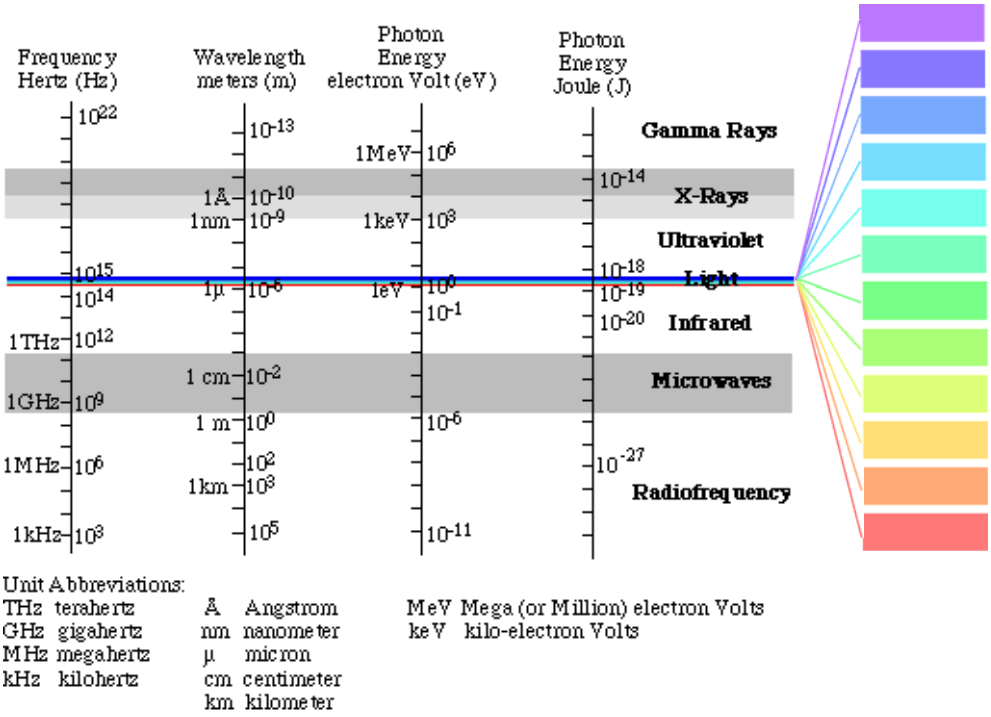


$$H(s) = \frac{X}{F} = \frac{1}{ms^2 + bs + k}$$

$$= \frac{1/m}{s^2 + \frac{b}{m}s + \frac{k}{m}} = \frac{1/m}{s^2 + 2\varsigma\omega_n s + \omega_n^2}$$

These results agree with intuition. The system acts as a "low-pass" filter with the break frequency growing smaller as the mass increases.

The mass associated with electronic polarization is that associated with electrons. Electrons are light—much lighter than nucleons. The mass associated with an electron is $9.11(10^{-31})$ kg. The mass of a protons and neutrons are approximately $1.67(10^{-27})$ kg. Electronic polarization is a mechanism that survives into the ultraviolet regions and does not begin dropping until the visible spectrum, which *starts at $4(10^{14})$ Hz.*



EM Spectrum from **NASA*

i) *electronic polarization* refers to the relative movement of a molecule's electron cloud with respect to its nuclei.

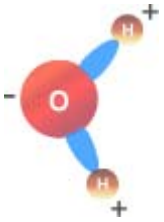
This is the mechanism behind rainbows and is what allows prisms to separate white light into a spectrum of colors. This is why focusing and other "optical processes become difficult for x-rays. At x-ray frequencies, all polarization mechanisms are long gone, which creates challenges for focusing, magnifying etc.

ii) *ionic polarization* refers to the relative movement of negative and positive ions in an ionic solid.

The mass associate with this mechanism is on the order of nuclei ~ typically around 10^4 or a few 10^4 times the mass of a single electron. This implies a break frequency two or three decades lower than that for electronic polarization.

This is just what is typically seen—*ionic polarization begins leaving in the vicinity of infrared (IR) frequencies ~ beginning around 10^{12} to 10^{13} Hz*.

iii) orientational polarization—associated with the movement of the permanent electric dipole of a polar molecule. This mechanism begins leaving in the microwave regions (a few GHz).



Water is a good example of the frequency dependence of orientational polarization and its subsequent affect on permittivity as frequency varies. At DC, water has a relative permittivity of 80 and at 100 GHz, its relative permittivity is around 10.

Dielectric strength and dielectric breakdown ☼

A related property is the material's **dielectric strength**. A material's dielectric strength is the maximum electric field that the material can withstand without damage. At sufficiently strong electric fields, electrons break their bonds, become free, and then accelerate due to the applied field. At this point the material no longer is an effective insulator.

Moreover, these electrons can, in turn, collide with other bound electrons, breaking their bonds. Once their bonds are broken, these newly free electrons accelerate under the strong electric field and, when they collide with a neighboring molecule, other electrons become free. This chain reaction can very quickly create large numbers of free electrons.

The process is referred to as **dielectric breakdown**. The heat generated by these energetic scattering events usually results in catastrophic damage to the dielectric.

material	**dielectric strength, K (V/m)**
air	$3(10^6)$
glass	$30(10^6)$
quartz	$40(10^6)$
polystyrene	$50(10^6)$
mica	$200(10^6)$

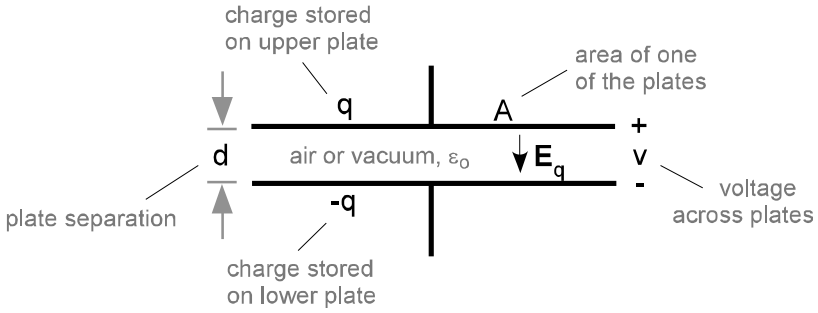
Capacitance

A capacitance exists between any two conductors separated by an insulator. The term capacitor comes from its "capacity" to store electric charge.

The capacitance of a capacitor is depends on the geometry of the conductors and the physical properties of the insulator, in particular its permittivity. The capacitance of a capacitor is equal to the ratio of the charge on the plates to the voltage across its plates.

Below is a parallel plate capacitor with air or vacuum as the insulator.

$$C = \frac{q}{V}$$

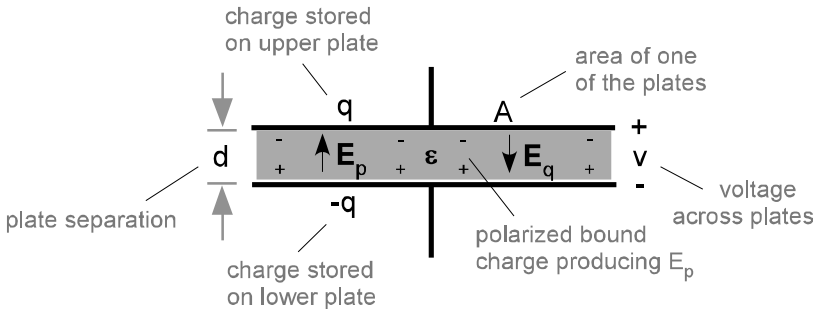


In circuits, the capacitor is thought of in terms of its current-voltage relation,

$$\frac{dq}{dt} = \frac{d}{dt}(CV) = C\frac{dV}{dt} \qquad \rightarrow \qquad i = C\frac{dV}{dt}$$

where C is assumed to be independent of time.

What happens when some material other than air or vacuum is placed between capacitance plates? The material will polarize in response to the electric field, which will act to reduce the total electric field.



Since the voltage between the plates is the line integral of the total electric field, V will be reduced for a given charge, q, stored on the plates. The presence of a dielectric increases the capacitor's capacitance compared to that when air or vacuum is the insulator.

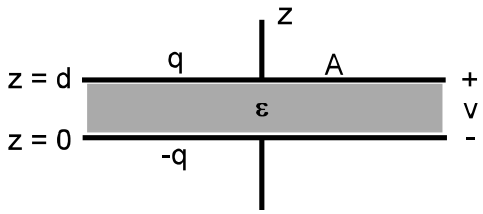
Calculation of capacitance

For the calculation of capacitance, Gauss' law ($\psi = q$) is used to express the capacitance in terms of electric flux rather than charge.

$$C = \frac{q_{\substack{\text{stored}\\ \text{on conductor}}}}{V_{\substack{\text{between}\\ \text{conductors}}}} = \frac{\psi_{\substack{\text{from}\\ \text{conductor}}}}{V_{\substack{\text{between}\\ \text{conductors}}}} = \frac{\iint_{\substack{\text{conductor}\\ \text{surface}}} \mathbf{D} \cdot d\mathbf{s}}{\int_{\substack{\text{between}\\ \text{conductors}}} \mathbf{E} \cdot d\mathbf{l}}$$

Example-parallel plate capacitor

The assumption in the parallel plate model is that the plate separation is considered sufficiently small to be negligible compared to plate area.



The plate area is very large so that the fields in the interior can be assumed to be those that would be obtained from infinite planes of charge. Then, if the plate area is large compared to the plate separation, the region in which these approximations do not hold (which would be along the edges, where the plates do not appear as approximately infinite) is small compared to the total plate area. These small effects, the "fringing fields" or "edge effects", are neglected in the parallel plate model.

This assumption allows one to consider the charge to be evenly distributed on the plates and to consider the field lines to run straight between the plates.

Solution outline

The surface charge density on the upper plate can be calculated.

$$\rho_s = \frac{q}{A}$$

Assuming the plates to be PECs, the electric flux density vector and the electric field vector can be calculated between the plates using Gauss' law or using results obtained when discussing boundary conditions.

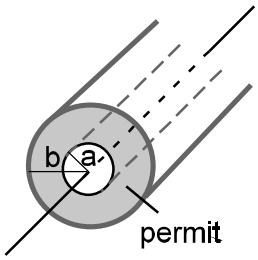
$$\mathbf{D} = \rho_s \left(-a_z\right) = \frac{q}{A}\left(-a_z\right) \qquad \rightarrow \qquad \mathbf{E} = \frac{q}{A\varepsilon}\left(-a_z\right)$$

Knowing the electric field, the potential difference, V, can be found from a line integral.

$$V = \int_{z=d}^{0} \mathbf{E} \cdot d\mathbf{l} = \int_{z=d}^{0} \frac{q}{A\varepsilon}\left(-\mathbf{a}_z\right) \cdot dz\, \mathbf{a}_z = -\frac{q}{A\varepsilon}\left(z\Big|_d^0\right) = \frac{qd}{A\varepsilon}$$

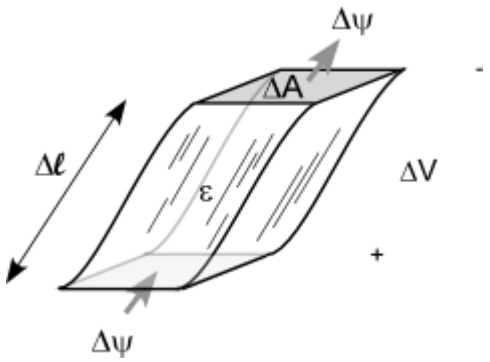
$$C = \frac{q}{V} = \frac{q}{qd/A\varepsilon} = \varepsilon\frac{A}{d}$$

Example—cylindrical capacitor ☼



Incremental capacitor

The element relationship for the incremental capacitor is the ratio of the charge (using Gauss, equivalent to electric flux) to potential difference.



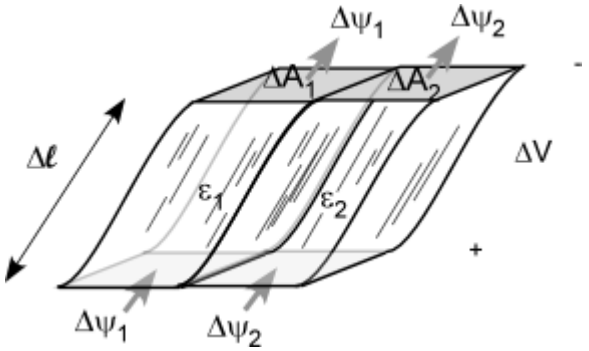
$$\Delta C = \frac{\Delta\psi}{\Delta V} = \frac{D\,\Delta A}{E\,\Delta \mathbf{l}} = \frac{\varepsilon E\,\Delta A}{E\,\Delta \mathbf{l}}$$

$$\Delta C = \frac{\varepsilon\,\Delta A}{\Delta \mathbf{l}}$$

The incremental capacitor provides an excellent means by which to explore the reasons behind the differences in how resistances and capacitances combine in series and in parallel.

Let's start with the incremental capacitor. Suppose two are in parallel with their ends on the same equipotential surfaces. How do they combine? Individually, we have,

Capacitors in parallel



$$\Delta C_1 = \frac{\varepsilon_1\,\Delta A1}{\Delta \mathbf{l}_1} \quad \text{and} \quad \Delta C_2 = \frac{\varepsilon_2\,\Delta A_2}{\Delta \mathbf{l}_2}$$

what is their combined capacitance?

$$\Delta C = \frac{\Delta\psi}{\Delta V} = \frac{D_1\,\Delta A_1 + D_2\,\Delta A_2}{E\,\Delta \mathbf{l}} = \frac{\varepsilon_1 E\,\Delta A_1 + \varepsilon_2 E\,\Delta A_2}{E\,\Delta \mathbf{l}}$$

$$\Delta C = \frac{\varepsilon_1\,\Delta A_1 + \varepsilon_2\,\Delta A_2}{\Delta \mathbf{l}} = \Delta C_1 + \Delta C_2$$

Capacitance in parallel add, just as we knew they did.

Capacitors in series

Now, how about capacitors in series?

$$\Delta C = \frac{\Delta\psi}{\Delta V} = \frac{D\,\Delta A}{E_1\,\Delta \mathbf{l}_1 + E_2\,\Delta \mathbf{l}_2} = \frac{D\,\Delta A}{\frac{D}{\varepsilon_1}\Delta \mathbf{l}_1 + \frac{D}{\varepsilon_1}\Delta \mathbf{l}_2}$$

$$\Delta C^{-1} = \frac{\frac{\Delta \mathbf{l}_1}{\varepsilon_1} + \frac{\Delta \mathbf{l}_2}{\varepsilon_2}}{\Delta A} = \frac{\Delta \mathbf{l}_1}{\varepsilon_1\,\Delta A} + \frac{\Delta \mathbf{l}_2}{\varepsilon_2\,\Delta A} = \Delta C_1^{-1} + \Delta C_2^{-1}$$

Compare these rules for combining capacitances to those for combining resistances?

Capacitance	Resistance
Equivalent capacitance for parallel capacitances is the sum of the individual capacitances.	Equivalent resistance for parallel resistances is the reciprocal of the sum of the reciprocals.
Equivalent capacitance for series capacitances is the reciprocal of the sum of the reciprocals.	Equivalent resistance for series resistances is the sum of the individual resistances.

What are the reasons behind these differences. The reasons are in the definitions of resistance and capacitance. Resistance is potential difference over flux (current) while capacitance is flux (electric flux) over potential difference.

$$R = \frac{\text{potential difference}}{\text{flux (current)}} \qquad C = \frac{\text{flux (electric flux)}}{\text{potential difference}}$$

The flux for elements in parallel add and the potential difference is common. For capacitance, the flux is in the numerator and the potential difference is in the numerator. Therefore, capacitances in parallel add. For resistance, the flux (current) is in the denominator and the potential difference is in the numerator. Therefore, the reciprocal of the individual resistances add, their sum being the reciprocal of the equivalent resistance.

The potential for elements in series add and their flux is common. For resistance, the potential is in the numerator and the flux is in the numerator. Therefore, resistances in series add. For capacitance, the potential is in the denominator and the flux is in the numerator. Therefore, the reciprocal of the individual capacitances can be added, their sum being the reciprocal of the equivalent capacitance.

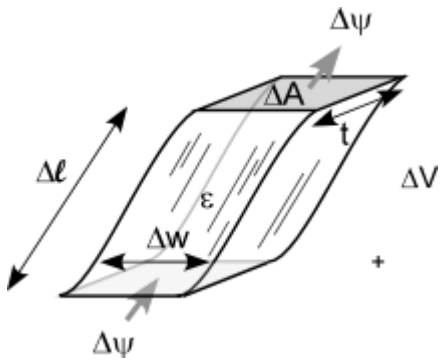
Per-square capacitance

Suppose the incremental capacitance is two-dimensional in character with a thickness t.

In this case, $\Delta C = \dfrac{\varepsilon \Delta A}{\Delta \mathbf{l}} = \dfrac{\varepsilon t \Delta w}{\Delta \mathbf{l}}$

Suppose further, that $\Delta \mathbf{l} = \Delta w$, in which case

$C_{\square} = \Delta C = \varepsilon t$

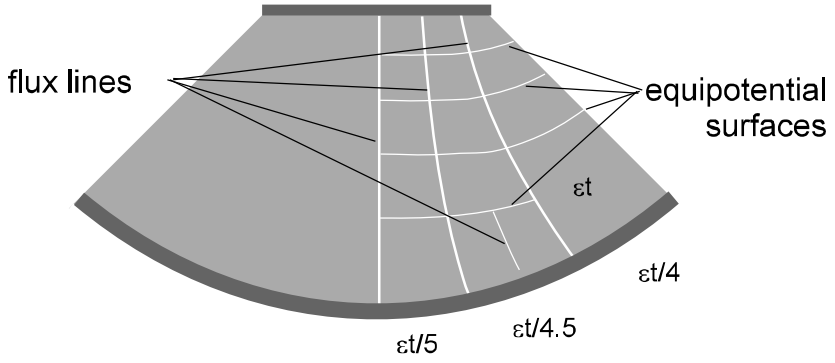


Example: curvilinear squares for irregular 2D capacitors

As for resistances, this approach can be generalized for capacitances with general cross-sections. For irregular shapes perfect squares are not possible, and "curvilinear squares" having an average width equal to an average height are used.

The boundaries of these "curvilinear squares" are chosen so that their sides are parallel to the flux and their ends are equipotential surfaces.

Assuming a thickness t, use curvilinear squares to graphically estimate the capacitance of the 2D capacitance shown.



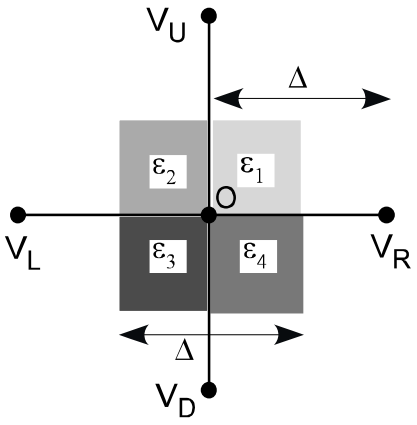
Symmetry can be used to reduce the work to two identical capacitances in parallel. It can be seen that the resistance of the 2D resistance is

$$C = 2\,\varepsilon t\left(5^{-1} + 4.5^{-1} + 4^{-1}\right) \cong 1.34\,\varepsilon t$$

General case of non-homogeneous permittivity

Case three: Boundaries neither parallel nor perpendicular to electric flux lines.

Consider two-dimensional capacitors in (xyz). The general cell is shown below.



The basic relation used is Gauss' law assuming an uncharged dielectric. Consequently, each region must satisfy $\psi_{out} = 0$.

Use the formula.

$$\psi = \mathbf{D} \bullet \text{area} = \varepsilon E \text{ (area)}$$

Looking at the right surface.

$$\psi_{out\ (right)} = \frac{V_O - V_R}{\Delta} \left\{ \left(\varepsilon_1 \frac{\Delta\ t}{2} \right) + \left(\varepsilon_4 \frac{\Delta\ t}{2} \right) \right\}$$

$$\psi_{out\ (right)} = \frac{V_O - V_R}{2} t \left(\varepsilon_1 + \varepsilon_4 \right)$$

Performing similar calculations at the upper, the left, and the bottom surfaces, we would obtain.

$$\psi_{out} = \frac{V_O - V_R}{2} t \left(\varepsilon_1 + \varepsilon_4 \right) + \frac{V_O - V_U}{2} t \left(\varepsilon_1 + \varepsilon_2 \right) +$$
$$\frac{V_O - V_L}{2} t \left(\varepsilon_2 + \varepsilon_3 \right) + \frac{V_O - V_B}{2} t \left(\varepsilon_3 + \varepsilon_4 \right) = 0$$

Rearranging, one obtains,

$$2 V_O \left(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 \right) - V_R \left(\varepsilon_1 + \varepsilon_4 \right) - V_U \left(\varepsilon_1 + \varepsilon_2 \right) - V_L \left(\varepsilon_2 + \varepsilon_3 \right) - V_B \left(\varepsilon_3 + \varepsilon_4 \right) = 0$$

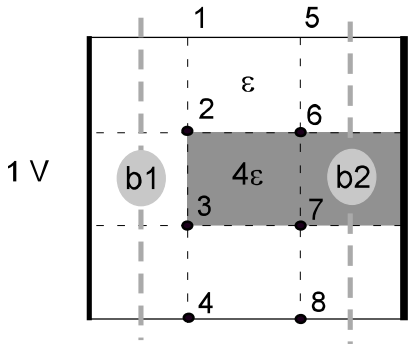
Solving for V_o,

$$V_O = \frac{V_U \left(\varepsilon_1 + \varepsilon_2 \right) + V_L \left(\varepsilon_2 + \varepsilon_3 \right) + V_B \left(\varepsilon_3 + \varepsilon_4 \right) + V_R \left(\varepsilon_1 + \varepsilon_4 \right)}{2 \left(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 \right)}$$

This is the equation that can be used to find the potentials in a two-dimensional inhomogeneous capacitor. The 2D case is importance since the majority of capacitors are in integrated circuits, the vast majority of which are 2D capacitors fabricated with planar deposition processes.

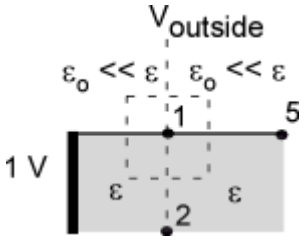
As with the resistive case, this technique can readily be generalized to the general inhomogeneous 3D capacitor.

Consider the two dimensional capacitor below. The permittivity of the shaded regions is 4ε and that of the unshaded regions is ε.



The solution will involve writing 8 node equations. Begin with N1.

$$4V_1 = 1\text{ V} + V_5 + 2V_2$$



The other 7 node voltage equations,

n2: $14V_2 = 2(1V) + 2V_1 + 5V_6 + 5V_3$

n3: $14V_3 = 2(1V) + 5V_2 + 5V_7 + 2V_4$

n4: $4V_2 = 1V + 2V_3 + V_8$

n5: $4V_5 = 1\text{ V} + 2V_6$

n6:

n7:

n8:

To find capacitance, divide total electric flux by the applied voltage.

$$C = \frac{\psi}{1\text{ V}} = \varepsilon t \left[\frac{(1 - V_1)}{2} + (1 - V_2) + (1 - V_3) + \frac{(1 - V_4)}{2} \right] \qquad @\ b1$$

OR

$$C = \frac{\psi}{1\text{ V}} = \varepsilon t \left(\frac{V_5}{2} + 2.5\,V_6 + 2.5\,V_7 + \frac{V_8}{2} \right) \qquad @\ b2$$

Mutual capacitance (may be omitted without loss of continuity)*

Given N bodies with q_n charge, superposition leads one to conclude that the potential function is a linear combination of the q_n charges. This section can be useful when using EM solvers since many use this approach; knowing the notation involved can avoid confusion.

$$V(\mathbf{r}) = \begin{cases} \int_{q'} \dfrac{dq'}{4\pi\varepsilon R} & \text{distribution} \\ \sum_{i=1}^{N} \dfrac{q_i}{4\pi\varepsilon R_i} & \text{discrete charges} \end{cases}$$

The potentials of the N bodies can likewise be expressed as linear combinations of the q_n.

$$\begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_{N-1} \\ V_N \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1N-1} & p_{1N} \\ p_{21} & p_{22} & & & \vdots \\ \vdots & & \ddots & & \\ p_{N-11} & & & & \\ p_{N1} & \cdots & & & p_{NN} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_{N-1} \\ q_N \end{bmatrix}$$

This relation can be inverted.

$$\begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_{N-1} \\ q_N \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1N-1} & c_{1N} \\ c_{21} & c_{22} & & & \vdots \\ \vdots & & \ddots & & \\ c_{N-11} & & & & \\ c_{N1} & \cdots & & & c_{NN} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_{N-1} \\ V_N \end{bmatrix}$$

Note, from physical grounds, that the diagonal elements, the c_{ii}, must be positive and that the off-diagonal elements, the c_{ij}, must be negative. This relation can be used to form expressions for our familiar capacitances. For example:

$$q_1 = (c_{11} + c_{12} + \cdots + c_{1N})V_1 + (-c_{12})V_{12} + \cdots + (-c_{1N})V_{1N}$$

Using capital letters to denote the the conventional capacitances, expressions can be obtained for the charge on a body in terms of body's potential and the potential difference between it and the other N-1 bodies. *Note all the capital C's are positive.*

$$q_1 = C_{11}V_1 + C_{12}V_{12} + \cdots + C_{1N}V_{1N}$$

*from Plonsey and Collins, "Principles and Applications of Electromagnetic Fields."

Energy density

Coulomb's law has been used to show the electrostatic potential energy in the interaction between two charges is

$$W_e = \frac{q\,q'}{4\pi\varepsilon\,R}$$

Where R is the distance between the charges. For multiple charges, the energy can be expressed as

$$W_e = \sum_{i<j} \frac{q_i\,q_j}{4\pi\varepsilon\,R_{ij}} = \frac{1}{2}\sum_{i\neq j} \frac{q_i\,q_j}{4\pi\varepsilon\,R_{ij}}$$

In terms of distributions,

$$W_e = \frac{1}{2}\iiint_r \iiint_{r'} \frac{\rho(\mathbf{r})\rho(\mathbf{r'})}{4\pi\varepsilon R}\,dv\,dv'$$

$$W_e = \frac{1}{2}\iiint_r \rho(\mathbf{r})V(\mathbf{r})\,dv = \frac{1}{2}\iiint_r (\nabla \cdot \mathbf{D})\,V(\mathbf{r})\,dv$$

using the vector identity $\nabla \cdot (V\mathbf{D}) = V(\nabla \cdot \mathbf{D}) + \mathbf{D} \cdot (\nabla V)$

$$W_e = \frac{1}{2}\iiint_r (\nabla \cdot \mathbf{D})\,V(\mathbf{r})\,dv = \frac{1}{2}\iiint_r \nabla \cdot (V\mathbf{D})\,dv - \frac{1}{2}\iiint_r \mathbf{D} \cdot (\nabla V)\,dv$$

using the divergence theorem for the first integral,

$$W_e = \frac{1}{2}\oiint_{surface} V\mathbf{D} \cdot d\mathbf{s} - \frac{1}{2}\iiint_r \mathbf{D} \cdot (\nabla V)\,dv$$

for $\rho \neq 0$ only within a finite region, then $V \to 0$ at least as $\frac{1}{r}$ and $D \to 0$ at least as $\frac{1}{r^2}$ which results in the first integral going to zero as $r \to \infty$.

$$W_e = -\frac{1}{2}\iiint_r \mathbf{D} \cdot (\nabla V)\,dv = \frac{1}{2}\iiint_r \mathbf{D} \cdot \mathbf{E}\,dv$$

$$w_e = \frac{dW_e}{dv} = \frac{1}{2}(\mathbf{D} \cdot \mathbf{E})$$

where w_e is the **electrostatic energy density**.

$$w_e = \frac{1}{2}(\mathbf{D} \cdot \mathbf{E})$$

Other forms given that $\mathbf{D} = \varepsilon\mathbf{E}$,

$$w_e = \frac{1}{2}\varepsilon E^2 = \frac{1}{2\varepsilon}D^2$$

Example: calculation of capacitance using energy

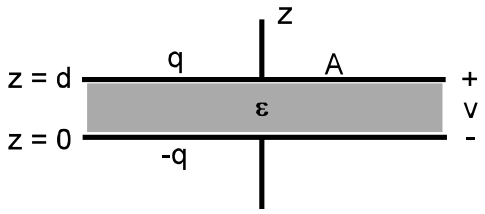
Energy storage provides another path by which capacitance can be calculated. From electrical circuits,

$$W_e = \frac{1}{2}CV^2 \qquad \rightarrow \qquad C = \frac{2\,W_e}{V^2}$$

In terms of fields,

$$C = \frac{2\iiint_{\text{volume}} \frac{1}{2}(\mathbf{D}\cdot\mathbf{E})dv}{\left(\int_{\substack{\text{between}\\\text{plates}}} \mathbf{E}\cdot d\mathbf{l}\right)^2} = \frac{\iiint_{\text{volume}} (\mathbf{D}\cdot\mathbf{E})dv}{\left(\int_{\substack{\text{between}\\\text{plates}}} \mathbf{E}\cdot d\mathbf{l}\right)^2}$$

Taking the parallel-plate capacitor as an example where the fields have already been calculated using the parallel plate model assumptions and Gauss' law.



$$\mathbf{D} = \frac{q}{A}(-a_z) \quad \text{and} \quad \mathbf{E} = \frac{q}{A\varepsilon}(-a_z)$$

$$C = \frac{\iiint_{\text{volume}} (\mathbf{D}\cdot\mathbf{E})dv}{\left(\int_{\substack{\text{between}\\\text{plates}}} \mathbf{E}\cdot d\mathbf{l}\right)^2} = \frac{\left[\frac{q}{A}(-\mathbf{a}_z)\cdot\frac{q}{A\varepsilon}(-\mathbf{a}_z)\right]Ad}{\left(\int_{z=d}^{0}\frac{q}{A\varepsilon}(-a_z)\cdot dz\,\mathbf{a}_z\right)^2} = \frac{\frac{q^2}{A^2\varepsilon}Ad}{\left(\frac{q}{A\varepsilon}d\right)^2}$$

$$C = \frac{\varepsilon A}{d}$$

Energy and force ☼

If W is the total system potential energy and is a function of some dimension l.

$$dW = -\mathbf{F} \cdot d\mathbf{l}$$

The force can be seen as a change in potential energy per unit length. The direction of the force is determined from whether energy increases or decreases with the direction of movement. The force exerted by the field is in the direction of energy decrease. As discussed in lecture, $\mathbf{F} = -\nabla W$.

Consider the parallel plate capacitance. Since the plates are oppositely charged, it is evident that there will be an electrostatic attraction between the plates—after all opposite charges attract.

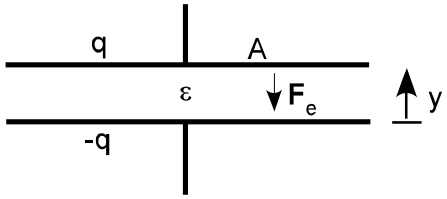
Two cases will be considered. The first is the case of constant stored charge during movement. The second is the case of the charge stored on the plates re We can find it by using this idea of energy.

$$W_e = \frac{1}{2} C V^2 = \frac{1}{2} C \left(\frac{q}{C}\right)^2 = \frac{q^2}{2C}$$

Case one is the case where the charge stored does not change. The electrical energy stored changes as the plates move.

$$W_e = \frac{q^2}{2C} = \frac{1}{2}\frac{y}{\varepsilon A} q^2$$

$$dW_e = \frac{1}{2}\frac{q^2}{\varepsilon A} dy$$



The electrical force exerted by the electrodes is,

$$\mathbf{F}_e = -\nabla W_e = -\left(\mathbf{a}_x \frac{\partial}{\partial x} + \mathbf{a}_y \frac{\partial}{\partial y} + \mathbf{a}_z \frac{\partial}{\partial z}\right)\frac{q^2}{2\varepsilon A} y = \frac{q^2}{2\varepsilon A}(-\mathbf{a}_y)$$

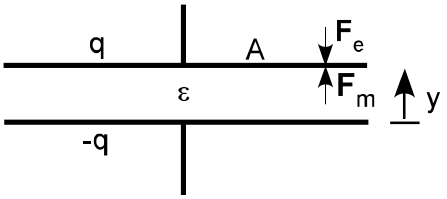
Another look: Consider this same problem, this time from the viewpoint of energy conservation. Consider the capacitor to be a system. With constant charge on the plates, energy energy can enter the system via the mechanical force, where the mechanical force ($\mathbf{F}_m$) must just balance the electrical force ($\mathbf{F}_e = -\mathbf{F}_m$) between the plates.

$$dW_e = \frac{1}{2} q\, dV = \frac{1}{2} q^2\, d\left(\frac{1}{C}\right) = F_m dy$$

$$\frac{q^2}{2\varepsilon A} dy = F_m\, dy$$

$$\mathbf{F}_e = -\mathbf{F}_m = -\mathbf{a}_y \frac{q^2}{2\varepsilon A}$$

,



Case two is when the capacitor is connected to an independent voltage source. In this case, since the voltage remains constant, the best relation to use for the energy stored by the capacitance is

$$dW_e = dW_{source} + dW_{cap}$$

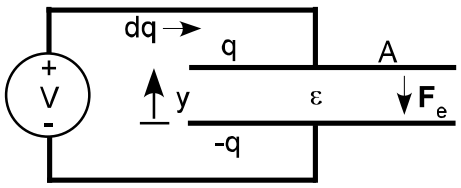
Also, the system includes a voltage source as well as the capacitor, so that the potential energy is the potential energy expression should include the charge supplied by the source.

$$dW_e = -V\,dq + \frac{1}{2}V\,dq = -\frac{1}{2}V\,dq$$

$$dW_e = -\frac{1}{2}V\,d(CV) = -\frac{1}{2}V^2\,d\left(\frac{\varepsilon A}{y}\right)$$

$$dW_e = \frac{1}{2}\frac{\varepsilon A V^2}{y^2}\,dy$$

$$\mathbf{F} = -\frac{dW_e}{dy}\mathbf{a}_y = -\frac{\varepsilon A V^2}{2y^2}\mathbf{a}_y$$



Another look: Consider this same problem, this time from the viewpoint of energy conservation. Consider the capacitor to be a system. Energy can enter the system in two ways—from the voltage source and from the mechanical force, where the mechanical force ($\mathbf{F}_m$) must just balance the electrical force ($\mathbf{F}_e = -\mathbf{F}_m$) between the plates.

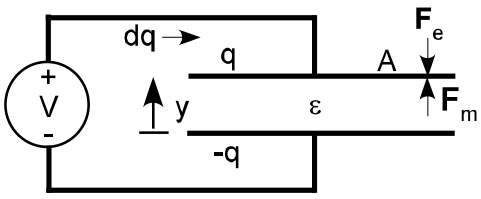
$$dW_{cap} = dW_{source} + dW_{mechl}$$

$$\frac{1}{2}V\,dq = V\,dq + F_m\,dy$$

$$-\frac{1}{2}V\,dq = F_m\,dy$$

$$-\frac{1}{2}V\,dq = -\frac{1}{2}V\,d(CV) = -\frac{1}{2}V^2\,d\left(\frac{\varepsilon A}{y}\right) = F_m\,dy$$

$$\frac{1}{2}\frac{\varepsilon A V^2}{y^2}\,dy = F_m\,dy \quad \Rightarrow \quad \mathbf{F}_e = -\mathbf{F}_m = -\mathbf{a}_y\frac{1}{2}\frac{\varepsilon A V^2}{y^2}$$

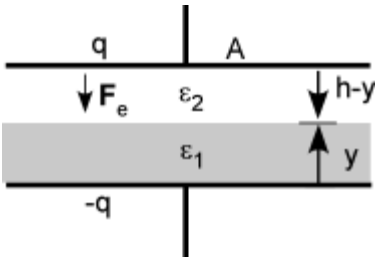


<u>The same reasoning can be used to analyze the pressures at interfaces.</u>

Interface normal to field lines. Assuming that the interface is uncharged,

$$D_{N1} = D_{N2}$$

Therefore, as the interface is moved the flux density and total flux remain unchanged which implies the charge q is constant.



$$W_e = \frac{q^2}{2C_1} + \frac{q^2}{2C_2} = \frac{q^2 y}{2\varepsilon_1 A} + \frac{q^2 (h-y)}{2\varepsilon_2 A}$$

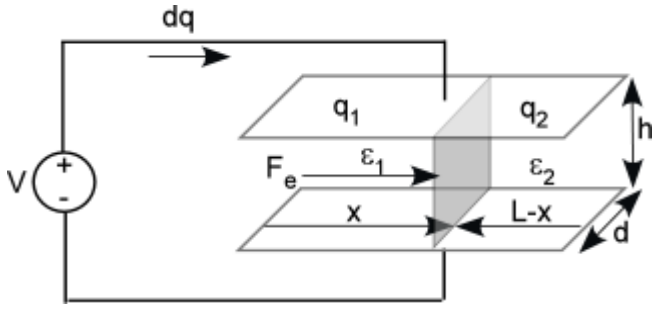
$$\mathbf{F}_e = -\nabla W_e = -\frac{q^2}{2\varepsilon_1 A}\mathbf{a}_y + \frac{q^2}{2\varepsilon_2 A}\mathbf{a}_y = \frac{q^2}{2A}\mathbf{a}_y\left(\frac{1}{\varepsilon_2} - \frac{1}{\varepsilon_1}\right)$$

In terms of energy densities,

$$\mathbf{F}_e = \mathbf{a}_y A\left(\frac{q^2}{2\varepsilon_2 A^2} - \frac{q^2}{2\varepsilon_1 A^2}\right) = \mathbf{a}_y A\left(\frac{D^2}{2\varepsilon_2} - \frac{D^2}{2\varepsilon_1}\right)$$

$$\mathbf{P}_e = \frac{\mathbf{F}_e}{A} = \mathbf{a}_y\left(\frac{D^2}{2\varepsilon_2} - \frac{D^2}{2\varepsilon_1}\right)$$ force is always directed from large to smaller ε.

Interface tangential to field lines. In this case, the electric field doesn't change and, since the plate separation is also constant, the voltage is constant.



$$dW_e = -V(dq_1 + dq_2) + \frac{1}{2}V\,dq_1 + \frac{1}{2}V\,dq_2 = -\frac{V^2}{2}(dC_1 + dC_2)$$

$$dW_e = -\frac{V^2}{2}\left[d\left(\varepsilon_1\frac{xd}{h}\right) + d\left(\varepsilon_2\frac{(L-x)d}{h}\right)\right] = -\frac{V^2 d}{2h}(\varepsilon_1 - \varepsilon_2)dx$$

$$\mathbf{F}_e = -a_x\frac{dW_e}{dx} = \mathbf{a}_x\frac{V^2 d}{2h}(\varepsilon_1 - \varepsilon_2)$$

In terms of energy densities,

$$\mathbf{F}_e = \mathbf{a}_x\frac{V^2 hd}{2h^2}(\varepsilon_1 - \varepsilon_2) = \mathbf{a}_x\ \frac{1}{2}(\varepsilon_1 - \varepsilon_2)\left(\frac{V}{h}\right)^2\ hd$$

$$\mathbf{P}_e = \frac{F_e}{hd} = \mathbf{a}_x\ \frac{1}{2}(\varepsilon_1 - \varepsilon_2)E^2$$

Physics of forces involving dielectrics

1. Electric pressure is directed from higher permittivity to lower permittivity. (The material with the higher permittivity tends to expand at the expense of the material with lower permittivity,)

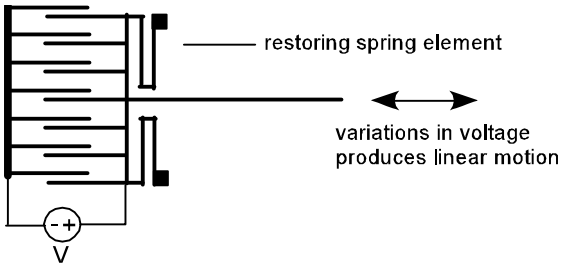
2. The boundary pressure at the interface is equal to the difference in energy densities.

Other applications and to look further

Link to ES applications such as electrostatic separation, xerography, laser printing, lightning rods, ion thruster, and electrophoresis. http://www.electrostatic.com Link to simulations of electrostatic applications using CST. http://www.cst.com/Content/Applications/Index/All+Electrostatic+Applications

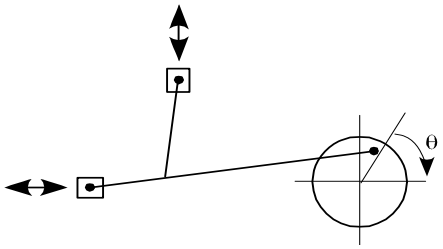
Electrostatic actuation and sensing

The above discussion provides some background for electric actuation. How is this put to use? **Comb drives** use electrostatic actuation, coupled with spring elements to produce **linear motion** by varying a voltage.



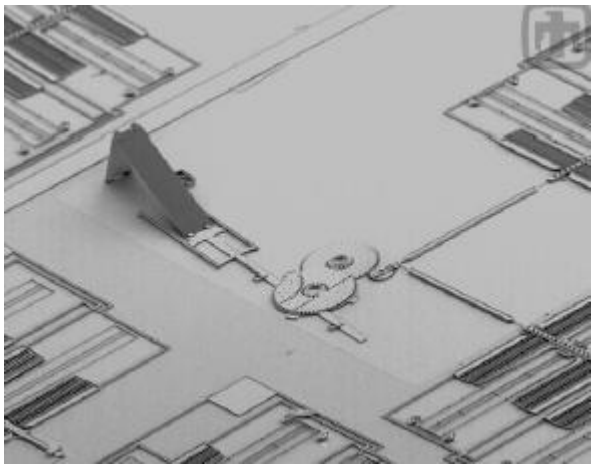
COMB DRIVE

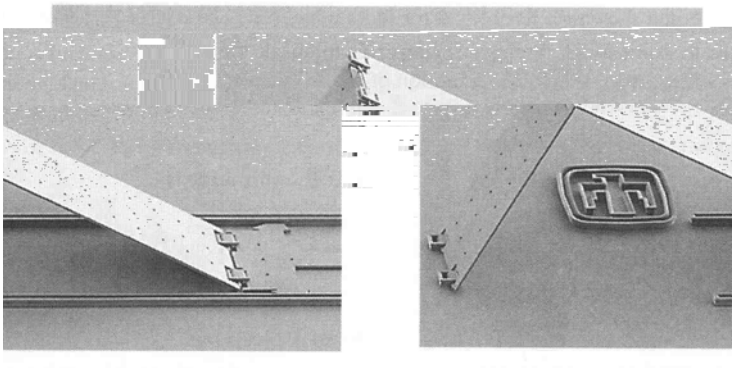
Comb drives, with gears, can produce **rotary motion**, a **micro-engine**.



Why are two comb drives necessary in the micro-engine?

Sandia's website shows devices giving **out-of-plane motion**.
http://www.sandia.gov/mstc/technologies/micromachines/movies/index.html

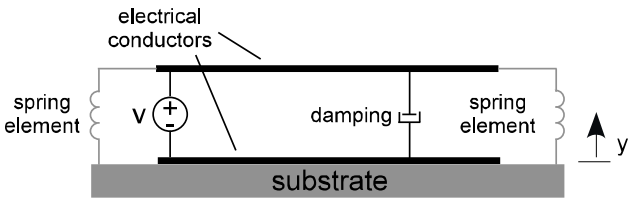


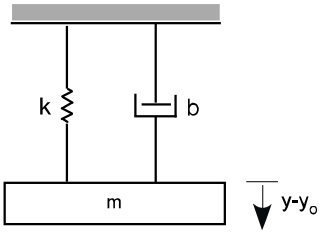


PHOTOGRAPHS FROM SANDIA NATIONAL LABORATORIES

Exercise: Pull-in

A well-known phenomenon of electrostatic actuators is pull-in, in which the two plates suddenly come together. Consider the static case and explore pull-in. Work with your neighbors to develop and discuss a physical explanation (y_0 is equilibrium point for V = 0). (about 5 minutes)

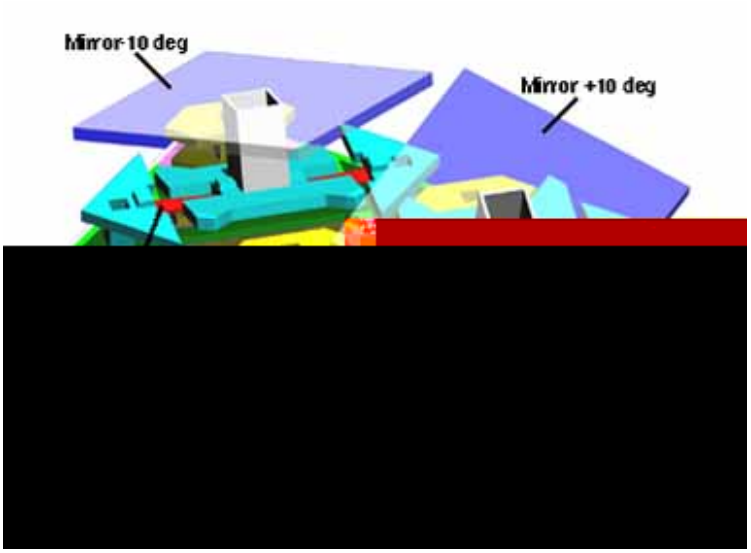




$$e = - y \frac{1}{2} \frac{A}{y^2} V^2$$

TI mirror array

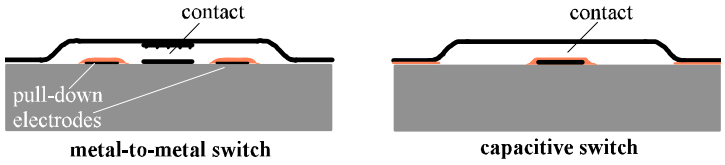
The technology behind DLP projection TVs are electrostatically driven micromirrors. Over a million of these micromirrors, each tilting over 5000 times per second enable applications in digital light processing (DLP).



TEXAS INSTRUMENTS MICRO-MIRROR http://www.dlp.com/

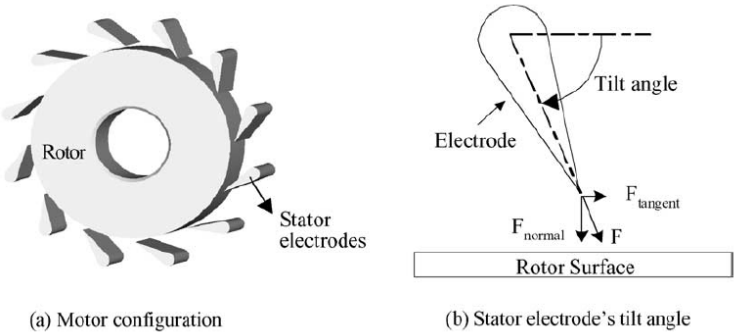
RF MEMS Switches

At high-frequencies (10-100 GHz), MEMS switches can achieve a more ideal open state that can solid state switches.



RF MEMS: THEORY, DESIGN, AND TECHNOLOGY (BY GABRIEL M. REBEIZ, PUBL. WILEY)

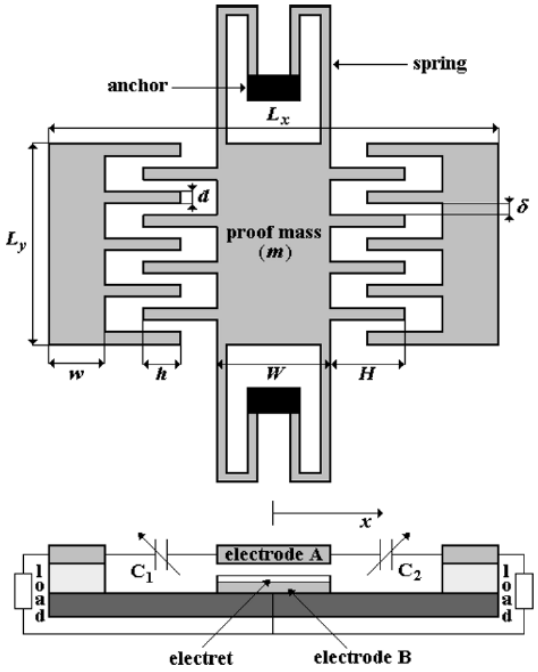
Corona Motor



(a) Motor configuration (b) Stator electrode's tilt angle

CORONA MOTOR (FROM *SENSORS AND ACTUATORS* **A118**, 226-332, 2005)

Energy Scavenging



ELECTROLET ENERGY SCAVENGING (PEANO AND TAMBOSSO, IEEE J. OF MEMS 14, 429-435 (2005)

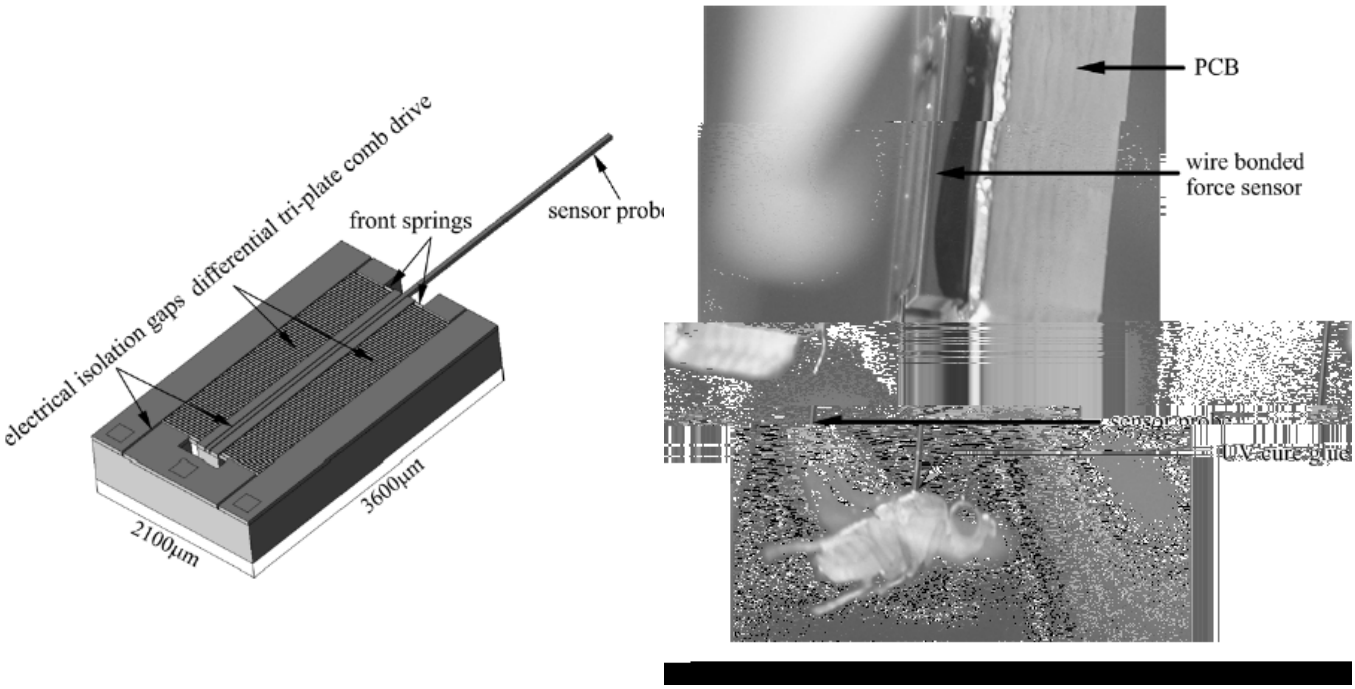
Sensors

Capacitive sensors change their capacitance due to either changes in geometry (plate separation, plate area) or/and due to changes in permittivity.

TABLE OF CAPACITIVE SENSORS

Capacitive Sensors	Principle of Operation	Measurement Uses
Pressure	Geometry varies with pressure	Capacitive pressure sensors are typically formed by forming closed cavities which contain capacitor conductors—typically one located on a membrane exposed to pressure to be measured. Varying the pressure varies the plate separation and so varies the capacitance.
Displacement	Geometry varies with displacement	One plate is usually fixed with the other movable. Capacitance displacement sensors are made which have resolutions of less than 10 pm
Accelerometer	Geometry varies with acceleration	Vibration control in hard-disk drives. Vibration detection in various consumer products. MEMS-based capacitive accelerometers are widely used to deploy automobile air bags.

Fruit Fly Sensor



FRUIT FLY ELECTROSTATIC COMB SENSOR (SUN ET AL., IEEE J. OF MEMS **14**, 4-11 (2005)