MA/CSSE 474 Exam 1 Winter 2009-10 Name____Solution______ Total score ____/120

SHORT_ANSWER QUESTIONS

1) (12 points) For each of the following statements, circle T or F to indicate whether it is *True* or *False*. If it is sometimes False, you should choose False. You do not have to give proofs or counterexamples. For each part, you get 1 point for leaving it blank, 2 for circling the correct answer, and 0 for circling the incorrect answer. Reason: Knowing that you don't know something counts for something.

a) T F $\forall L\ ((L^+)^+ = L^+)$.

True. .

b) T F $\forall L_1, L_2, L_3\ ((L_1L_2L_3)^* = L_1{}^*L_2{}^*L_3{}^*)$.

False.

c) T F $\forall L\ (L^*\ L = L^+)$.

True.

d) T F $\forall L\ (\varnothing\ L^* = \varnothing)$.

True.

e) T F $\forall L_1, L_2, L_3\ (((L_1\ L_2) \cup (L_1\ L_3))^* = (L_1\ (L_2 \cup L_3))^*)$.

True.

f) T F The set INF (all infinite languages) is closed under the "complement" function.

False

2. (5 points) Describe integer exponentiation n^m as a language recognition problem.

Problem: Given two nonnegative numbers *n* and *m*, compute n^m.

Encoding of the problem: We transform the problem of computing n^m into the problem of checking whether a third number *p* is the result of raising *n* to the *m* power.

The language to be decided: INTEGEREXP =

{*w* of the form: <$integer_1$>**<$integer_2$>=<*integer*3> : each of the substrings <*integer*1>,

<*integer*2> and <*integer*3> is an element of $[0 - 9]^+$ and *integer*3 = $integer_1{}^{integer2}$.

3. (8 points) Recall the function *chop*(*L*) defined in Example 4.10.

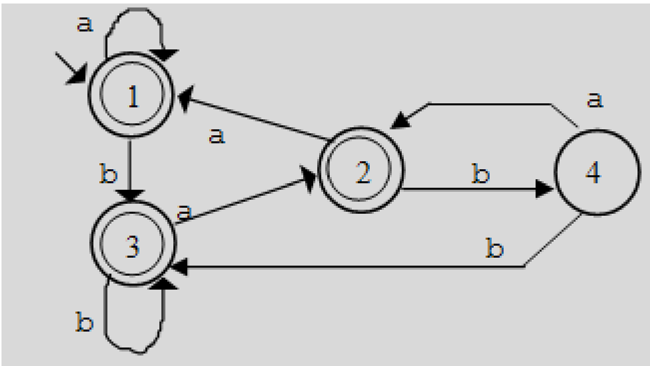
$chop(L) = \{w : \exists x \in L\ (x = x_1cx_2,\ x_1 \in \Sigma_L{}^*,\ x_2 \in \Sigma_L{}^*,\ c \in \Sigma_L,\ |x_1| = |x_2|, \text{ and } w = x_1x_2)\}$.

What is $chop(\{a^nb^{2n} : n \geq 0\})$?

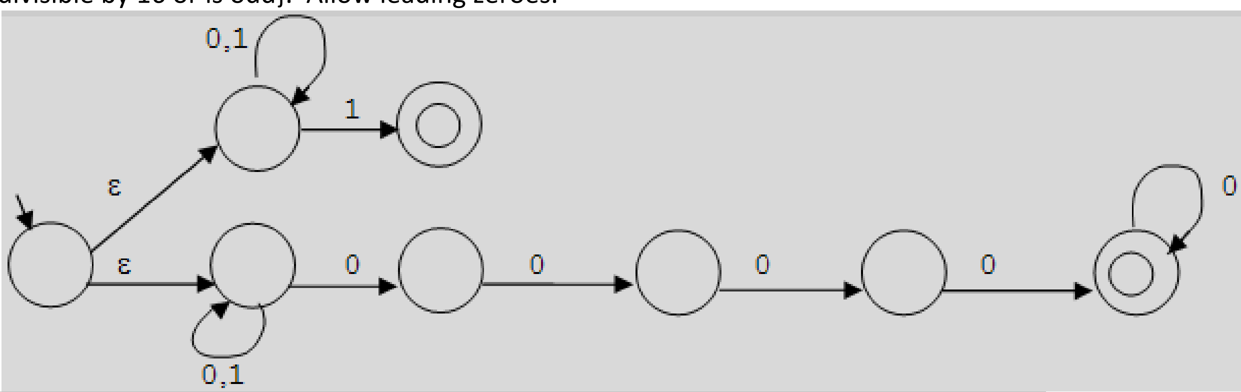
$\{a^nb^{2n-1} : n \geq 1$ and odd$\}$. Another way to describe it: $\{a^{2n+1}b^{4n+1} : n \geq 0\}$

PROBLEMS

4. Draw the transition diagrams for FSMs that accept the following languages:
   a) (15 points) {*w* ∈ {a, b}* : *w* does not end in bab}.
   Make the machine for "ends with bab", then switch A with K – A.

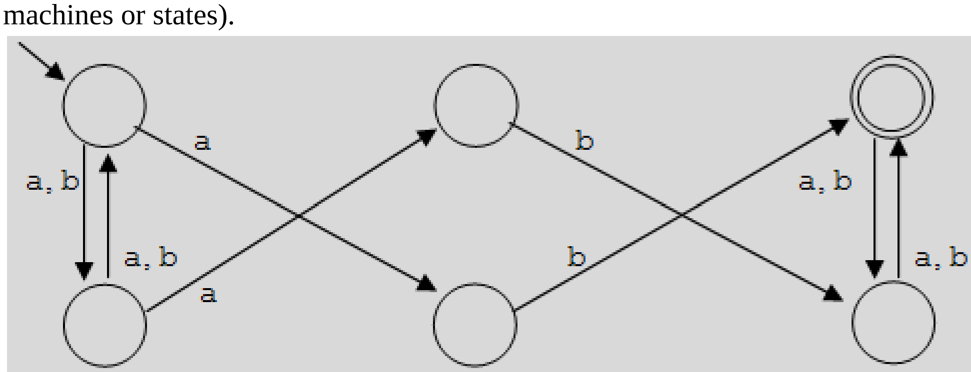


   b) (15 points) {*w* ∈ {0, 1}* : *w* is the binary encoding (most significant digit first) of a positive integer that is divisible by 16 or is odd}. Allow leading zeroes.



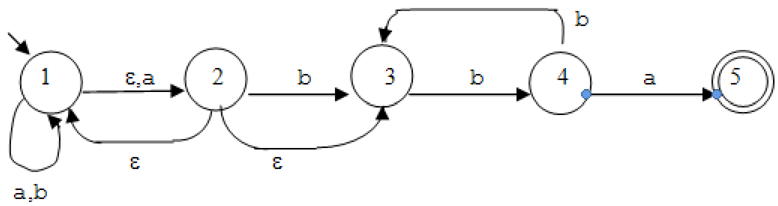
This one allows leading zeroes. A machine that does not will also be considered correct.

5. (20 points) If M is the FSM represented by the diagram below, describe L(M) in English (without using references to machines or states).



Answer: {*w* ∈ {a, b}* : *w* has even length and contains the substring ab}

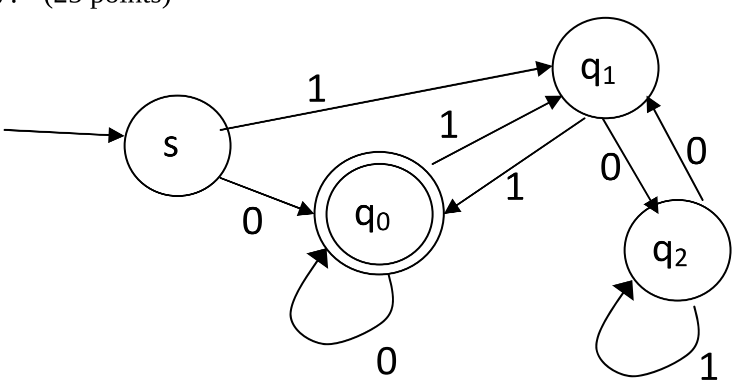
6. (20 points) For each the following NDFSM, use *ndfsmtodfsm* to construct an equivalent DFSM. Begin by showing the value of *eps*(q) for each state q:



eps(1) = {1, 2, 3}
eps(2) = {2, 3, 1}
eps(3) = {3}
eps(4) = {4}
eps(5) = {5}

δ({1, 2, 3}, a) = {1, 2, 3}
δ({1, 2, 3}, , b) = {1, 2, 3, 4}
δ({1, 2, 3, 4}, a) ={1, 2, 3, 5}   State{1, 2, 3, 5} is the accepting state.
δ({1, 2, 3, 4}, b) ={1, 2, 3, 4}
δ({1, 2, 3, 5}, a) ={1, 2, 3}
δ({1, 2, 3, 5}, b) = {1, 2, 3, 4}

7. (25 points)



Consider the DFSM M shown above. Σ={0, 1}.
Let L = {w∈Σ* : w is a binary encoding of an integer that is divisible by 3 (leading zeroes allowed) }.
Prove (using Mathematical induction) the following Theorem: L = L(M) , the language accepted by M.

That is, prove that $\forall w \in \Sigma^*((s, w) \vdash_M^* (q_0, \varepsilon) \leftrightarrow w \in L)$

As in the long in-class proof of NDFSMtoDFSM equivalence, you can't directly use induction to prove the theorem, because the property is not necessarily true of shorter strings. So you need a lemma that is a more general property. That lemma can be proved by induction. Once you have proved the lemma, the proof of the theorem is simple.
If you cannot figure out what the lemma should be, **I will "sell" you its statement for 5 points.** If you cannot prove the lemma, you can use the lemma to prove the theorem and earn 5 points. Summary: Come up with the lemma = 5 points, prove the lemma = 15 points, use the lemma to prove the theorem = 5 points.

☐ Bought the lemma (Instructor use)

Note: Some part of the proof is going to involve a case for each of q_0, q_1, and q_2. You are allowed to explain one case in detail, to say that the other case is similar, and only write how the details of the other cases differ from the one you do in detail.
Continue your proof on the back of the page if you wish.

Proof: I will write much more here than I expect you to write on the exam. On the exam, you are to show me that you understand the proof. Here, I attempt to write enough detail to help those who don't understand it to get to a point of understanding.

Special case: $w = \varepsilon$. Clearly $w \notin L$. And $(s, w) \vdash_M^* (s, \varepsilon)$, $s \neq q_0$.

Otherwise: If w is any nonempty string in Σ^*, there is some $r \in \mathbb{N}$ such that $w = <r>$ (binary representation of r).

Lemma: Let % be the normal remainder operator, as in C, Java, or Python.

If w is any nonempty string in Σ^*,

$\forall r \in \mathbb{N}, k \in \{0, 1, 2\}\ ([w = <r> \wedge r \% 3 = k] \leftrightarrow [(s, w) \vdash_M^* (q_k, \varepsilon)])$

Proof of Lemma: By induction on $|w|$.

Base case: $|w| = 1$. There are two possibilities:

w=0. $0 = <0>$. And by the transition diagram, $(s, 0) \vdash_M^* (q_0, \varepsilon)$. ✓

w=1. $1 = <1>$. And by the transition diagram, $(s, 1) \vdash_M^* (q_1, \varepsilon)$. ✓

Induction step: (if part) $|w| = n+1$ for some $n \in \{1, 2, 3, \ldots\}$. Assume by induction that the lemma is true for strings of length n, and show that it is true for w. Note from the diagram that once the machine enters one of the sates in $\{q_0, q_1, q_2\}$, it will subsequently always be in one of those three states. So any string of length ≥ 1 takes us from s to one of those three steps. Since $|w| \geq 1$, $w=<r>$ for some $r \in \mathbb{N}$

Since $|w| = n+1$, $w = xa$ for some $a \in \Sigma$, $x \in \Sigma^*$, $|x| = n$.

Since $|x| \geq 1$, $x=<s>$ for some $s \in \mathbb{N}$.

For some $k \in \{0, 1, 2\}$, $s \% 3 = k$. Note that this means **$s = 3m + k$** for some $m \in \mathbb{N}$.We consider the 3 cases:

Case 1: k=1. By induction, $(s, x) \vdash_M^* (q_1, \varepsilon)]$. What is $\delta(q_1, a)$?

If a=0, $\delta(q_1, a)= q_2$. Thus $(s, w) = (s, xa) \vdash_M^* (q_1, a) \vdash_M (q_2, \varepsilon)$. We only need to show that $r \% 3 = 2$.

$<r>$ $= w = x0 = <s>0 = <2s> =<2(\mathbf{3m + 1})> = <3(2m) +2>$, so $r = 3(2m)+2$. ✓

If a=1, $\delta(q_1, a)= q_0$. Thus $(s, w) = (s, xa) \vdash_M^* (q_1, a) \vdash_M (q_0, \varepsilon)$. We only need to show that $r \% 3 = 0$.

$<r>$ $= w = x1 = <s>1 = <2s+1> =<2(\mathbf{3m + 1})+1> = <3(2m) +3> = <3(2m+1)>$, so $r = 3(2m)$. ✓

For the rest of the cases, you can say "similar to case 1", but since I can use copy and paste, I'll show the details.

Case 0: k=0. By induction, $(s, x) \vdash_M^* (q_0, \varepsilon)]$. What is $\delta(q_0, a)$?

If a=0, $\delta(q_0, a)= q_0$. Thus $(s, w) = (s, xa) \vdash_M^* (q_0, a) \vdash_M (q_0, \varepsilon)$. We only need to show that $r \% 3 = 0$.

$<r>$ $= w = x0 = <s>0 = <2s> =<2(\mathbf{3m + 0})> = <3(2m) >$, so $r = 3(2m)$. ✓

If a=1, $\delta(q_0, a)= q_1$. Thus $(s, w) = (s, xa) \vdash_M^* (q_0, a) \vdash_M (q_1, \varepsilon)$. We only need to show that $r \% 3 = 1$.

$<r>$ $= w = x1 = <s>1 = <2s+1> =<2(\mathbf{3m + 0})+1> = <3(2m) +1>$, so $r = 3(2m)+1$. ✓

Case 2: k=2. By induction, $(s, x) \vdash_M^* (q_2, \varepsilon)]$. What is $\delta(q_2, a)$?

If a=0, $\delta(q_2, a)= q_1$. Thus $(s, w) = (s, xa) \vdash_M^* (q_2, a) \vdash_M (q_1, \varepsilon)$. We only need to show that $r \% 3 = 1$.

$<r>$ $= w = x0 = <s>0 = <2s> =<2(\mathbf{3m + 2})> = <3(2m) + 4 > = <3(2m+1) + 1>$, so $r = 3(2m+1) + 1$. ✓

If a=1, $\delta(q_2, a)= q_2$. Thus $(s, w) = (s, xa) \vdash_M^* (q_2, a) \vdash_M (q_2, \varepsilon)$. We only need to show that $r \% 3 = 2$.

$<r>$ $= w = x1 = <s>1 = <2s+1> =<2(\mathbf{3m + 2})+1> = <3(2m) +5> = <3(2m+1) + 2>$, so $r = 3(2m)+2$. ✓

(only-if part) The arguments for each of the three cases are reversible, and these three cases are all there is.
For all positive integers n, if $n\%3 = k$, then the machine ends up in state q_k when processing the string that is $<n>$.

Use the lemma to prove the theorem:

If $w \in L(M)$, $(s, w) \vdash_M^* (q_0, \varepsilon)$. By the lemma, w is the binary representation of some r, where $r\%3 = 0$. So $w \in L$ ✓

If $w \in L$, then w is the binary representation of some r, where $r\%3 = 0$. By the lemma, $(s, w) \vdash_M^* (q_0, \varepsilon)$. Since q0 is an accepting state, $w \in L(M)$. ✓