MA/CSSE 474 Homework #0 Solution (and notes on grading - not graded Winter 2013-14)

A.1

$((A \wedge B) \rightarrow C) \leftrightarrow (\neg A \vee \neg B \vee C) \equiv$	
$(\neg(A \wedge B) \vee C) \leftrightarrow (\neg A \vee \neg B \vee C)$	Definition of $\rightarrow$
$((\neg A \vee \neg B) \vee C) \leftrightarrow (\neg A \vee \neg B \vee C)$	de Morgan's Law
$(\neg A \vee \neg B \vee C) \leftrightarrow (\neg A \vee \neg B \vee C)$	Associativity of $\vee$
True	Definition of $\leftrightarrow$

$(A \wedge \neg B \wedge \neg C) \rightarrow (A \vee \neg(B \wedge C)) \equiv$	
$\neg(A \wedge \neg B \wedge \neg C) \vee (A \vee \neg(B \wedge C))$	Definition of $\rightarrow$
$\neg A \vee B \vee C \vee (A \vee \neg(B \wedge C))$	de Morgan's Law
$\neg A \vee B \vee C \vee A \vee \neg(B \wedge C)$	Associativity of $\vee$
$\neg A \vee B \vee C \vee A \vee \neg B \vee \neg C$	de Morgan's Law
$\neg A \vee A \vee B \neg B \vee C \vee \neg C$	Commutativity of $\vee$
$(\neg A \vee A) \vee (B \neg B) \vee (C \vee \neg C)$	Associativity of $\vee$
True $\vee$ *True* $\vee$ *True*	Definition of $\vee$
True	Definition of $\vee$

A.2

a) $\varnothing$, {apple}, {pear}, {banana}, {apple, pear}, {apple, banana}, {pear, banana}, {apple, pear,banana}

b) {b}, {a, b}

c) $\varnothing$

d) There are none.

e) 7

f) 2, 3, 4, 5, 6, 7, 8, 9, 10, 11

g) 0, 1, 2, 3, 4, 5, 6, 7

(p-6) A.4

The first meaning says that, if the condition is true, then the only thing that is liked by everyone is tea:

$(\exists x, y\ (B(x) \wedge S_B(x) \wedge H(y) \wedge S_S(y)) \rightarrow \forall z\ (\neg T(z) \rightarrow \exists p\ (P(p) \wedge \neg L(p, z))))$

The second meaning says that, if the condition is true, then, for each individual person, the only thing that individual likes is tea:

$(\exists x, y\ (B(x) \wedge S_B(x) \wedge H(y) \wedge S_S(y)) \rightarrow \forall p, z\ (P(p) \wedge \neg T(z) \rightarrow \neg L(p, z))$.

There are other possibilities as well. The part before the arrow is non-controversial. Give credit for any reasonable interpretation of the part after the arrow. 4 points for the part before the arrow, 1 each for parts after arrow.

(p-4) A.5

$\forall x \in L\ (E(x, \mathtt{A}) \leftrightarrow (V(x) \wedge S(x, 3) \wedge \neg O(x, \mathtt{STUPID})))$.

$(\exists x \in L\ (B(\mathtt{K}, x) \wedge B(x, \mathtt{R}) \wedge S(x, 1))$.

Two points for each part, partial credit for reasonable attempts.

A.6

Let $A = \mathbb{N}$. Then write $\{x \in \mathbb{N} : x \neq 0 \wedge \sqrt{x} \in \mathbb{N}\}$.

A.7

The only such set is $\varnothing$.

A.8

Example: *Lives-within-a-mile-of*

A.9

Symmetric only.

(p-3-2-2) A.11

a) Yes, R_p is an equivalence relation for every $p \geq 1$:

- It is reflexive since, for all $a \in \mathbb{N}$, $a \equiv_p a$.
- It is symmetric since, for all $a, b \in \mathbb{N}$, $a \equiv_p b \rightarrow b \equiv_p a$.
- It is transitive since, for all $a, b, c \in \mathbb{N}$, $a \equiv_p b \wedge b \equiv_p c \rightarrow a \equiv_p c$. If there are integers r and s such that a – b = r*p and b – c = s*p, then a – c = (a – b) + (b – c) = (r +s)*p.

b)

There are p equivalence classes. The i^{th} equivalence class contains those elements that are equal to i modulo p.

c)

Neither. For R_p to be a partial order, it would have to be antisymmetric. Since it's symmetric, it can't be antisymmetric. Since every total order is also a partial order, R_p is also not total.

The 3-2-2 means 3 points for a, 2 for b, 2 for c. To get the 3 points of a, they should address R, S< and T individually.

A.12

Substr is a partial order. It is clearly reflexive and antisymmetric.
And Substr(x, y) $\wedge$ Substr(y, z) $\rightarrow$ Substr(x, z)

A.13

a) No. *father-of*(*John Quincy Adams*) = *John Adams* and *father-of*(*Charles Adams*) = *John Adams*.

b) No. There is no element x of P such that *father-of*(x) = *Abigail Adams*.

A.14

a) The negative integers are not closed under subtraction. For example (-4) – (-6) = 2. So the closure is the set of integers.

b) The negative integers are not closed under division. For example, -4/-2 = 2, so the positive integers must be added to the closure. But 2/4 is not an integer. So the closure is the set of rational numbers – {0}.

c) The positive integers are closed under exponentiation.

d) The finite sets are closed under cross product. Given two finite sets, x and y, $|x \times y| = |x| * |y|$.

(t-2) 1 point for correct answer (YES). 1 for a reasonable explanation.

e) The odd integers are not closed under remainder, mod 3. The range of this function is {0, 1, 2}. So the closure is the odd integers union with {0, 2}.

f) By construction. If x and y are rational, then they can be represented as a/b and c/d, where a, b, c, and d are integers. The sum of x and y is then:

$$\frac{ad+cb}{bd}$$

Since the integers are closed under both addition and multiplication, both the numerator and the denominator are integers. Since neither b nor d is 0, neither is the denominator. So the result is rational.

A.15

a) Let x be a natural number: $\{x \geq 0\} \cap \{x \leq 0\} = \{0\}$.

b) Let x be a natural number: $\{x \geq 0\} \cap \{x \geq 0\} = \{x \geq 0\}$.

c) Let x be a real number: $\{x \geq 0\} \cap \{x \leq 0\} = \{0\}$.

d) Let S_1 be the set of all positive real numbers. Let S_2 be the set of all negative real numbers union the set of all positive integers. Both S_1 and S_2 are uncountably infinite. But $S_1 \cap S_2$ = the set of all positive integers, which is countably infinite.

e) Let S be the set of real numbers. $S \cap S = S$.

(p-3) A.17

a) Unsatisfiable. The proof is by counterexample. If $x = 0$, there is no natural number y that is less than x.

b) Tautology. The proof is by construction. For any number x, one such y is $x + 1$.

c) Satisfiable but not a tautology. Let f be the identity function. Then the sentence is true. Let f be the function divide by 2. Then, for any odd value of x, there is no natural number y such that $f(x) = y$.

One point for each part. ½ for correct answer, ½ for proof. Of course students may have different examples than mine

A.18

We show a bijection f that maps from A to $\mathbb{N}$: $f(x) = x/3$.

A.19

a) (Same as for problem 18) We show a bijection f that maps from $\{n \in \mathbb{N} : n \equiv_3 0\}$ to $\mathbb{N}$: $f(x) = x/3$.

b) The only prime number that is evenly divisible by 3 is 3. So the cardinality of this set is 1.

c) Both $\{n \in \mathbb{N} : n \equiv_3 0\}$ and $\{n \in \mathbb{N} : n \text{ is prime}\}$ are countably infinite. The following algorithm constructs an infinite enumeration of their union U:

Consider the elements of $\mathbb{N}$ in order. For each do:

See if it is evenly divisible by 3.
See if it is prime.
If it passes either test, enumerate it.

We thus have an infinite enumerate of U. By Theorem A.1, a set is countably infinite iff there exists an infinite enumeration of it. So U is countably infinite.

(t-4) You could also just point to Theorem A.2. This earns full credit. But if you relied on that theorem, be sure that you understand how to prove it.

A.20

We show how to put the rational numbers in one-to-one correspondence with the natural numbers via a bijection we'll call f. We'll describe f as a function from the rationals to the natural numbers. Let $f(0) = 0$. Every nonzero rational number can be uniquely described as a sign bit plus a quotient p/q of two relatively prime positive integers. Consider the table shown below. Let the rows of the table correspond to p and the columns correspond to q. f will assign odd integers to the positive rationals by walking through the table, starting in the upper left corner and then following the arrows as shown. Whenever it encounters a cell that corresponds to a (p, q) pair where p and q are relatively prime, it will assign the next positive odd integer to the corresponding positive rational. f will assign even integers to the negative rationals in the same way.

	1	2	3	4	5	…
1						
2						
3						
…						

So we get: $f(0) = 0$. $f(1/1) = 1$. $f(2/1) = 3$. f assigns no value to the cell 2/2. $f(3/1) = 5$. $f(3/2) = 7$. And so forth. $f(-1/1) = 2$. $f(-2/1) = 4$. And so forth.