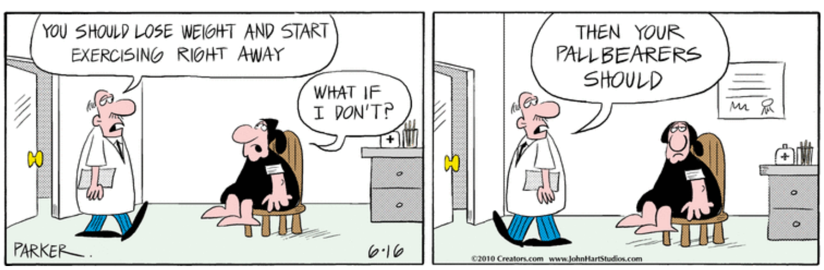
MA/CSSE 474 Theory of Computation

TM Variations
Encoding a TM
(Universal Turing Machine)

Your Questions?

- Previous class days' material
- Reading Assignments
- HW 14b problems
- Tuesday's exam
- Anything else



Two Flavors of TMs

1. Recognize a language
2. Compute a function

Turing Machines as Language Recognizers

Let $M = (K, \Sigma, \Gamma, \delta, s, \{y, n\})$.

- M ***accepts*** a string w iff $(s, \underline{\text{Æ}}w) \mid\!-_M^* \ (y, w')$ for some string w' (that includes an underlined character).
- M ***rejects*** a string w iff $(s, \underline{\text{Æ}}w) \mid\!-_M^* \ (n, w')$ for some string w'.

M ***decides*** a language $L \subseteq \Sigma^*$ iff:
For any string $w \in \Sigma^*$ it is true that:
if $w \in L$ then M accepts w, and
if $w \notin L$ then M rejects w.

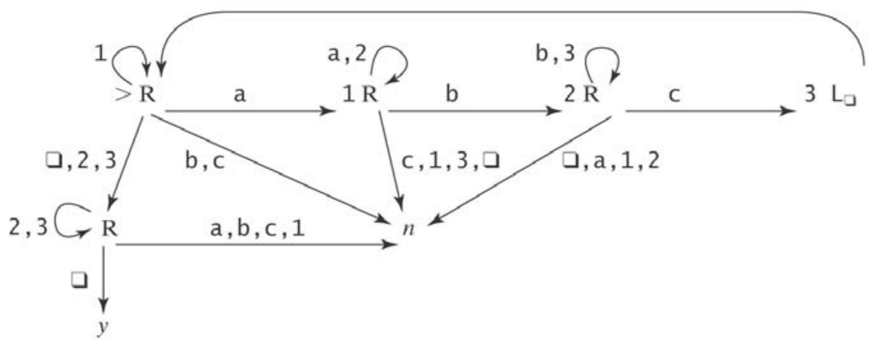
A language L is ***decidable*** iff there is a Turing machine M that decides it. In this case, we will say that L is in ***D***.

A Deciding Example

$A^nB^nC^n = \{a^nb^nc^n : n \geq 0\}$

Example: Æ aabbccÆ Æ Æ Æ Æ Æ Æ Æ Æ

Example: Æ aaccbÆ Æ Æ Æ Æ Æ Æ Æ Æ



Semideciding a Language

Let Σ_M be the input alphabet to a TM M. Let $L \subseteq \Sigma_M{}^*$.

M ***semidecides*** *L* iff, for any string $w \in \Sigma_M{}^*$:

- $w \in L \rightarrow M$ accepts w
- $w \notin L \rightarrow M$ does not accept w.

M may either:
reject or
fail to halt.

A language *L* is ***semidecidable*** iff there is a Turing machine that semidecides it. We define the set ***SD*** to be the set of all semidecidable languages.

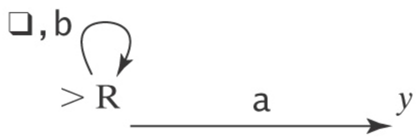
Example of Semideciding

Let L = b*a(a ∪ b)*

We can build M to semidecide L:

1. Loop
   1.1 Move one square to the right. If the character under the read head is an a, halt and accept.

In our macro language, M is:

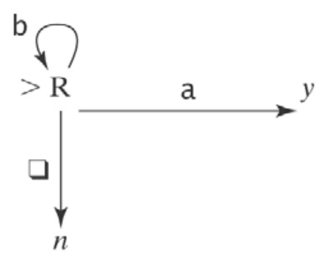


Example of Deciding the same Language

L = b*a(a ∪ b)*. We can also decide L:

Loop:
   1.1 Move one square to the right.
   1.2 If the character under the read/write head is an a, halt and accept.
   1.3 If it is Æ , halt and reject.

In our macro language, M is:



TM that Computes a Function

Let $M = (K, \Sigma, \Gamma, \delta, s, \{h\})$.

Define $M(w) = z$ iff (s, Æ w) $|\text{-}_M{}^*$ (h, Æ z).

Let $\Sigma' \subseteq \Sigma$ be M's output alphabet.
Let f be any function from Σ^* to Σ'^*.

Notice that the TM's function computes with strings (Σ^* to Σ'^*), not directly with numbers.

M ***computes*** function f iff, for all $w \in \Sigma^*$:

- If w is an input on which f is defined: $M(w) = f(w)$.
- Otherwise $M(w)$ does not halt.

A function f is ***recursive*** or ***computable*** iff there is a Turing machine M that computes it and that always halts.

Note that this is different than our common use of *recursive.*

Example of Computing a Function

Let $\Sigma = \{a, b\}$. Let $f(w) = ww$.

Input: Æ *w*Æ Æ Æ Æ Æ Æ Output: Æ *ww*Æ

Define the copy machine C:
Æ *w*Æ Æ Æ Æ Æ Æ → Æ *w*Æ *w*Æ

Also use the $S_{\leftarrow}$ machine:
Æ *u*Æ *w*Æ → Æ *uw*Æ

Then the machine to compute f is $>C\ S_{\leftarrow}\ L_{Æ}$

More details next slide

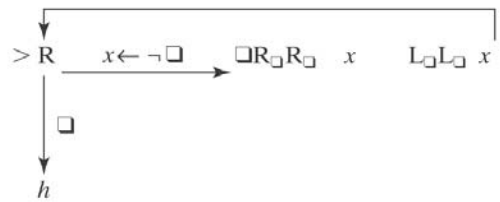
Example of Computing a Function

Let Σ = {a, b}. Let *f*(*w*) = *ww.*

Input: Æ *w*Æ Æ Æ Æ Æ Æ Output: Æ *ww*Æ

Define **the copy machine C**:

Æ *w*Æ Æ Æ Æ Æ Æ → Æ *w*Æ *w*Æ



Then use **the $S_{\leftarrow}$ machine**:

Æ *u*Æ *w*Æ → Æ *uw*Æ

Then the machine to compute *f* is >*C* $S_{\leftarrow}$ $L_{Æ}$

Computing Numeric Functions

For any positive integer *k*, the function ***value**$_k$**(n)*** returns the nonnegative integer that is encoded, base *k*, by the string *n*.

For example:

- *value*$_2$(`101`) = 5.
- *value*$_8$(`101`) = 65.

TM *M* **computes a function *f* from $\mathbb{N}^m$ to $\mathbb{N}$** iff, for some *k*:

$$value_k(M(n_1;n_2;\ldots n_m)) = f(value_k(n_1), \ldots\ value_k(n_m))$$

Note that the semicolon serves to separate the representations of the arguments

Why Are We Working with Our Hands Tied Behind Our Backs?

Turing machines — Are more powerful than any of the other formalisms we have studied so far.

☺

Turing machines — Are a **lot** harder to work with than all the real computers that are available to us.

☹

Why bother?

The very simplicity that makes it hard to program Turing machines makes it possible to reason formally about what they can do. If we can, once, show that *everything* a real computer can do can be done (albeit clumsily) on a Turing machine, then we have a way to reason about what real computers can do.

Multiple tracks

Multiple tapes

Non-deterministic

TURING MACHINE VARIATIONS

Turing Machine Variations

There are many extensions we might like to make to our basic Turing machine model. We can do this because:

We can show that every extended machine has an equivalent* basic machine.

We can also place a bound on any change in the complexity of a solution when we go from an extended machine to a basic machine.

Some possible extensions:

- Multi-track tape.
- Multi-tape TM
- Nondeterministic TM

Recall that *equivalent* means "accepts the same language," or "computes the same function."

Multiple-track tape

We would like to be able to have TM with a multiple-track tape. On an n-track tape, Track i has input alphabet Σ_i and tape alphabet Γ_i.

1st track	....	1	B	B	B	B	B	B	B	..
2nd track	....	B	B	1	B	B	B	B	B	..
3rd track	..	B	B	B	B	1	B	B	B	..
4th track	..	B	B	B	B	B	B	1	B	.
5th track	..	a	b	c	d	e	f	g	h	.

Multiple-track tape

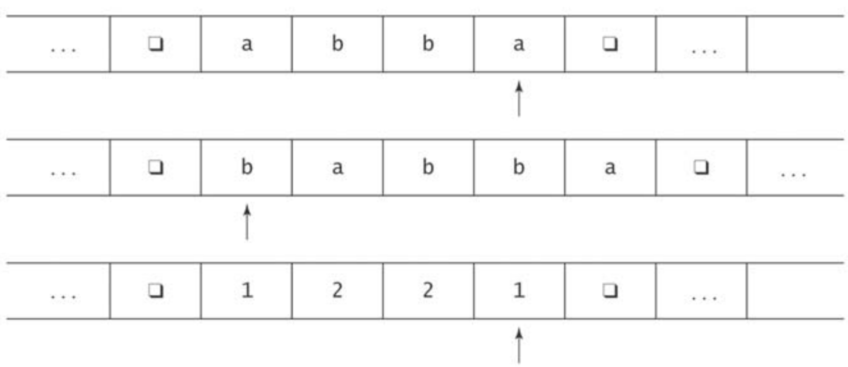
We would like to be able to have a TM with a multiple-track tape. On an n-track tape, Track i has input alphabet Σ_i and tape alphabet Γ_i.

We can simulate this with an ordinary TM.

A transition is based on the current state and the combination of all of the symbols on all of the tracks of the current "column".

Then Γ is the set of n-tuples of the form [$\gamma_1, \ldots, \gamma_n$], where $\gamma_1 \in \Gamma_i$. Σ is similar. The "blank" symbol is the n-tuple [❑, …, ❑]. Each transition reads an n-tuple from Γ, and then writes an n-tuple from Γ on the same "square" before the head moves right or left.

Multiple Tapes



Multiple Tapes

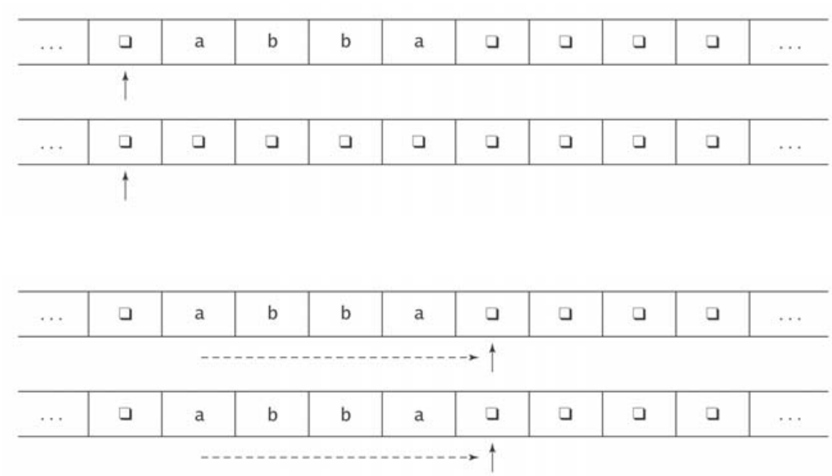
The transition function for a k-tape Turing machine:

$$
\begin{array}{llll}
((K\text{-}H), & \Gamma_1 & \text{to} & (K, \Gamma_{1'}, \{\leftarrow, \rightarrow, \uparrow\} \\
, & \Gamma_2 & & , \Gamma_{2'}, \{\leftarrow, \rightarrow, \uparrow\} \\
, & . & & , \; . \\
, & . & & , \; . \\
, & \Gamma_k) & & , \Gamma_{k'}, \{\leftarrow, \rightarrow, \uparrow\})
\end{array}
$$

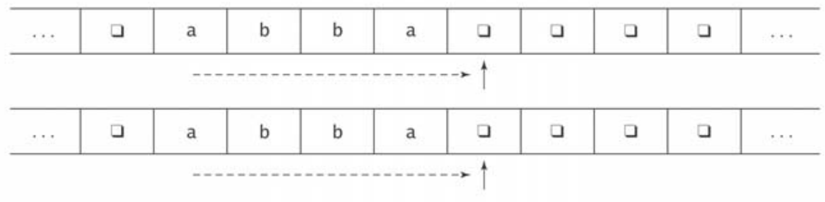
Input: initially all on tape 1, other tapes blank.
Output: what's left on tape 1, other tapes ignored.

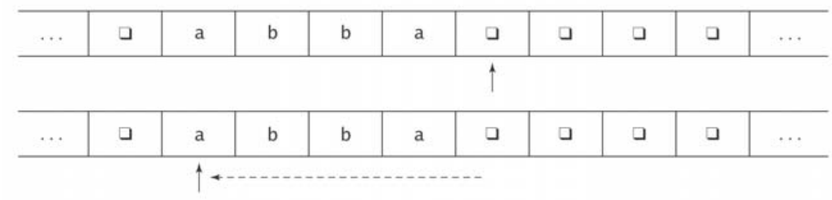
Note: On each transition, any tape head is allowed to stay where it is.

Example: Copying a String

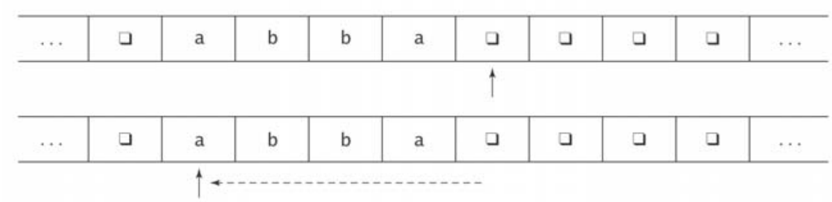


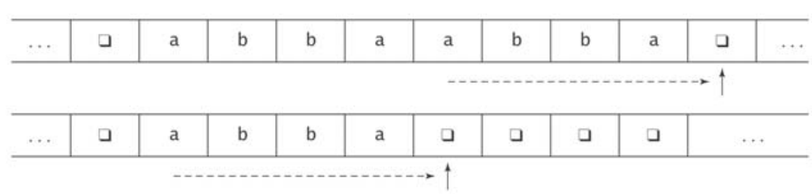
Example: Copying a String





Example: Copying a String





Another Two Tape Example: Addition

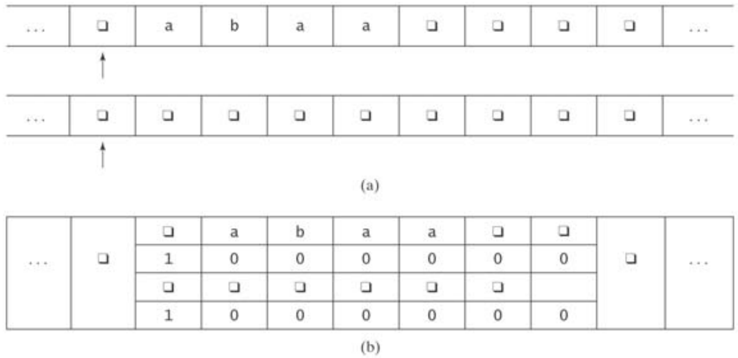
Adding Tapes Does Not Add Power

Theorem: Let $M = (K, \Sigma, \Gamma, \delta, s, H)$ be a k-tape Turing machine for some $k > 1$. Then there is a standard TM $M' = (K', \Sigma', \Gamma', \delta', s', H')$ where $\Sigma \subseteq \Sigma'$, and:

- On input x, M halts with output z on the first tape iff M' halts in the same state with z on its tape.
- On input x, if M halts in n steps, M' halts in $\mathcal{O}(n^2)$ steps.

Proof: By construction.

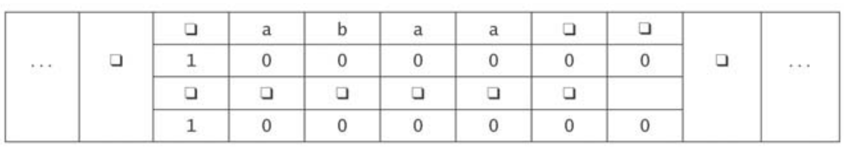
The Representation



Alphabet (Γ ') of *M'* = $\Gamma \cup (\Gamma \times \{0, 1\})^k$:

Æ , a, b, (Æ , 1, Æ , 1), (a, 0, Æ ,0), (b, 0, Æ , 0), …

The Operation of *M'*



1. Set up the multitrack tape.
2. Simulate the computation of *M* until (if) *M* would halt:
   2.1 Scan left and store in the state the *k*-tuple of characters under the read heads.
   Move back right.
   2.2 Scan left and update each track as required by the transitions of *M*. If necessary, subdivide a new (formerly blank) square into tracks.
   Move back right.
3. When *M* would halt, reformat the tape to throw away all but track 1, position the head correctly, then go to *M*’s halt state.

How Many Steps Does *M'* Take?

Let: w be the input string, and
n be the number of steps it takes M to execute.

Step 1 (initialization): $\mathcal{O}(|w|)$.

Step 2 (computation):
Number of passes = n.
Work at each pass: 2.1 = $2 \cdot$ (length of tape).
= $2 \cdot (|w| + n)$.
2.2 = $2 \cdot (|w| + n)$.
Total: $\mathcal{O}(n \cdot (|w| + n))$.

Step 3 (clean up): $\mathcal{O}$(length of tape).

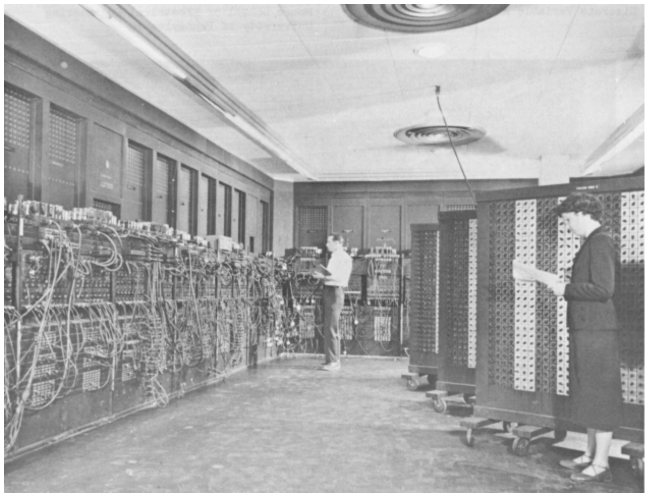
Total: $\mathcal{O}(n \cdot (|w| + n))$.
$= \mathcal{O}(n^2)$. *

* assuming that $n \geq w$

Universal Turing Machine

The Universal Turing Machine

Problem: All our machines so far are hardwired.



ENIAC - 1945

The Universal Turing Machine

Problem: All our machines so far are hardwired.

Question: Can we build a programmable TM that accepts as input:

program *input string*

executes the program on that input, and outputs:

output string

The Universal Turing Machine

Yes, it's called the ***Universal Turing Machine***.

To define the Universal Turing Machine *U* we need to:

1. Define an encoding scheme for TMs.

2. Describe the operation of *U* when it is given input <*M*, *w*>, the encoding of:

   - a TM *M*, and

   - an input string *w*.

Encoding the States

- Let i be $\lceil \log_2(|K|) \rceil$.
  Each state is encoded by a letter and a string of i binary digits.

- Number the states from 0 to $|K|$-1 in binary:
  - The start state, s, is numbered 0.
  - Number the other states in any order.

- If t' is the binary number assigned to state t, then:
  - If t is the halting state y, assign it the string yt'.
  - If t is the halting state n, assign it the string nt'.
  - If t is the halting state h, assign it the string ht'.
  - If t is any other state, assign it the string qt'.

Example of Encoding the States

Suppose *M* has 9 states.

i = 4

s = `q0000`,

Remaining states (suppose that *y* is 3 and *n* is 4):

```
q0001   q0010   y0011   n0100
q0101   q0110   q0111   q1000
```

Encoding a Turing Machine *M*, Continued

The tape alphabet:

Let *j* be $\lceil \log_2(|\ \Gamma\ |) \rceil$.
Each tape alphabet symbol is encoded as
a*y* for some $y \in \{0, 1\}^+$, $|y| = j$

The blank symbol gets the j-character representation of 0

Example: Γ = { Æ , a, b, c }. *j* = 2.

Æ = a00
a = a01
b = a10
c = a11

Encoding a Turing Machine *M*, Continued

The transitions: (state, input, state, output, move)

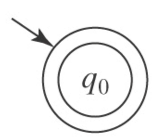
Example: (q000, a000, q110, a000, →)

Specify *s* as q000.

Specify *H*.

A Special Case

We will treat this as a special case:



An Encoding Example

Consider *M* = ({ *s*, *q*, *h* }, { a, b, c }, { Æ , a, b, c }, δ, *s*, { *h* }):

state	symbol	δ
s	Æ	(*q*, Æ , →)
s	a	(*s*, b, →)
s	b	(*q*, a, ←)
s	c	(*q*, b, ←)
q	Æ	(*s*, a,→)
q	a	(*q*, b, →)
q	b	(*q*, b, ←)
q	c	(*h*, a, ←)

state/symbol	representation
s	q00
q	q01
h	h10
Æ	a00
a	a01
b	a10
c	a11

Decision problem: Given a string w, is there a TM M such that w=<M> ?

Is this problem decidable?

<*M*> = (q00,a00,q01,a00,→), (q00,a01,q00,a10,→),
(q00,a10,q01,a01, ←), (q00,a11,q01,a10,←),
(q01,a00,q00,a01,→), (q01,a01,q01,a10,→),
(q01,a10,q01,a11,←), (q01,a11,h10,a01,←)