MA/CSSE 474 Exam 1 Winter 2013-14 Name____**SOLUTION**______ Section: 02 (3^{rd})___ 03(4^{th}) ___

Closed book and notes, except for one 8.5 x 11 sheet of paper (can be 2-sided).

Put your name on that paper and turn it in (in a separate pile) when you turn in your exam paper.

No electronic devices, especially anything with headphones or earbuds.

Scores:

Problem	Possible	Score
1	40	
2	15	
3	10	
4	10	
5	15	
6	10	
7	30	
Total	**130**	

1. (40 points) Circle T or F to indicate whether it is *True* or *False*. IDK means "I don't know"
If the statement is sometimes False, then False is the correct answer. **You need not give proofs or counterexamples.**
For each part, you earn **2 points** for circling IDK, **4** for circling the correct answer, **-2** for circling the incorrect answer, and **0** if you leave it blank. Leaving it blank is silly, since you get more points for IDK.

a) T F IDK Every alphabet is finite.

b) T F IDK The largest possible language is countably infinite.

c) T F IDK The set of all languages over alphabet {a} is countably infinite **Uncountable**

d) T F IDK INF (the set of infinite languages) is closed under complement.
{a, ab} is complement of an infinite language
I.e. the complement of an infinite language is infinite.

e) T F IDK Every regular language is finite. {a}* is regular.

f) T F IDK The complement of a regular language must be regular.

g) T F IDK For every language L, L∅L ⊆ L.

h) T F IDK For every language L, L ⊆ L∅L **L∅L = ∅**

i) T F IDK The power set of ∅ is ∅. **It is {∅}.**

j) T F IDK Recall that maxstring(L) = $\{w \in L: \forall z \in \Sigma^*(z \neq \varepsilon \rightarrow wz \notin L)\}$.
There is a language L such that maxstring(L) contains the empty string ε.

2. (15 points) Let $L \subseteq \Sigma^*$ for some alphabet Σ. In class we defined the relation $\approx_L$ on strings from Σ* to be
$x \approx_L y$ if and only if $\forall z \in \Sigma^*(xz \in L \leftrightarrow yz \in L)$. We showed that $\approx_L$ is an equivalence relation, and we used the notation [w] to mean "the equivalence class that contains w".

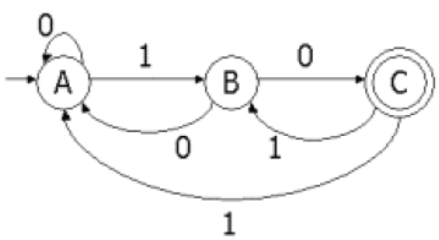
Carefully prove that for $\forall x,y \in \Sigma^*(\forall a \in \Sigma(([x] = [y]) \rightarrow ([xa] = [ya])))$. *This is NOT an induction proof.*

We did this proof in class last week.
[xa] = [ya] is a statement about all strings $z \in \Sigma^*$, so it is difficult to prove directly. The contrapositive of ([x] = [y]) → ([xa] = [ya]) is ([xa] ≠ [ya]) →([x] ≠ [y]); that is what we prove. **(You did not have to write this)**

If [xa] ≠ [ya], then there is a string z that distinguishes them, i.e (WLOG) xaz ∈ L and yaz ∉ L.

If we rewrite these as x(az) and y(az), then we see that az distinguishes x and y, so [x] ≠ [y].

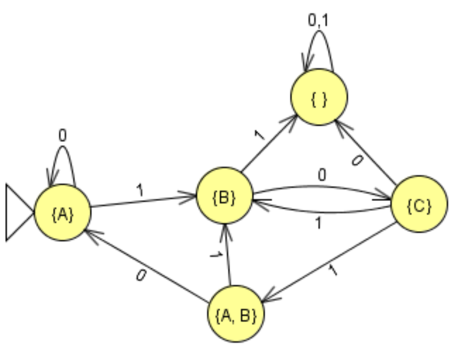
3. (10 points) The following nondeterministic finite state machine accepts which of the following strings? (Only one is correct).



- 1011101
- 1110100
- 1011010
- 00110100

The path through states is ABCABCBC

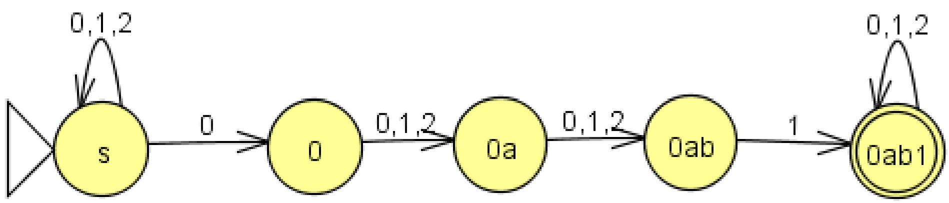
4. (10 points) If we run the *ndsfsmtodfsm* algorithm on the above NDFSM, which of the following potential states of the resulting DFSM is *not* reachable?



- {B}
- {A}
- {A,B,C}
- {}

5. (15 points) Let Σ = {0, 1, 2}. Let L be {w∈Σ* : ∃a,b∈Σ(∃x,y∈Σ*: w=x0ab1y}. I.e., each string contains a 1 and a 0, with two characters in-between. Draw the state diagram for a FSM M such that L = L(M).

My example is a nondeterministic machine.



6. (10 points) Let Σ={a}, and let C = {L⊆Σ* : (L=L(M) for some DFSM M) ∧ (M=(K, Σ, s, A, δ)) ∧ (|K|=2)}.

How many different languages are in C? __**6**__

If you get the correct number, full credit. Feel free to list the languages,, in case it helps you earn partial credit.
Let the states be s and p.
We enumerate the languages based on which states are accepting states:

s	p	Languages
no	no	∅
yes	no	{ε}, {a}*, {even-length strings of a's}
no	yes	∅, {odd-length strings of a's}, $\{a\}^+$
yes	yes	{a}*

7. (30 points) For this induction problem, you have a choice. You can do the first one to earn up to 30 points. Or you can do the simpler second one to earn up to 18 of the 30 possible points. If you write something for both parts, be sure to indicate which one you want me to grade. Whichever one you do, give the reasons for your steps, and in particular, make it clear where and how you use the induction hypothesis.

We have seen two different notations for a series of transitions by a DFSM $M=(K, \Sigma, s, A, \delta)$.

On Day 1, we saw the "extended delta" function. Here, I will call that function $\hat{\delta}$ to distinguish it from δ, since they take arguments of different types:

$(\hat{\delta} : K \times \Sigma^* \to K$, while $\delta: K \times \Sigma \to K)$.

Here is the recursive definition of $\hat{\delta}$:

$\hat{\delta}(q, \varepsilon) = q$,

$\hat{\delta}(q, wa) = \delta(\hat{\delta}(q,w), a)$ for all $w \in \Sigma^*$, $a \in \Sigma$.

The textbook's $\vdash^*$ can be defined recursively:

For all $w,t \in \Sigma^*$, $a \in \Sigma$,

$(q, t) \vdash^* (q, t)$

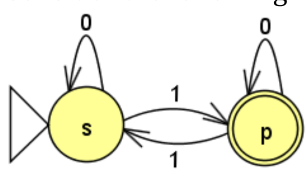
If $(q, wt) \vdash^* (p, t)$ for some $p \in K$ with $(p, a) \vdash (r, \varepsilon)$, then $(q, wat) \vdash^* (r, t)$

Prove (using induction on $|w|$ where needed)

$\forall q,r \in K, (\forall w \in \Sigma^* (\hat{\delta}(q,w) = r \leftrightarrow (q, w) \vdash^* (r, \varepsilon)))$.

I use the $\hat{\delta}$ notation (described in the other part of this problem, because I think it will make this proof easier to express. Feel free to use the textbook's $\vdash^*$ notation instead.

Consider the following DFSM:



Carefully prove, using mathematical induction on $|w|$ where needed, that $\hat{\delta}(s, w) = p$ if and only if w contains an odd number of 1's

First the → direction.

Base case: $w = \varepsilon$, so that $r = q$. By the base case of the definition of $\vdash^*$ (with the t in the definition being ε), $(q, \varepsilon) \vdash^*(q, \varepsilon)$.

Induction step. Let $w = xa$, and assume by induction that the property is true for the shorter string x. Then there must be state p such that $\hat{\delta}(q, x) = p$, and $\delta(p, a) = r$. By induction, $(q, xa) \vdash^*(p, a)$. Also $(p, a) \vdash (r, \varepsilon)$ [definition of $\vdash$]. Putting these last two things together, $(q, xa) \vdash^*((r, \varepsilon)$ [recursive definition of $\vdash^*$].

Next the ← direction.

Base case: $w = \varepsilon$, so that $r = q$, since a DFSM reading the empty string does not make a move. By the base case of the definition of $\hat{\delta}$, $\hat{\delta}(q, \varepsilon) = q = r$.

Induction step. Let $w = xa$, and assume by induction that the property is true for the shorter string x. Then there must be state p such that $(q, xa) \vdash^*(p, a)$ and $\delta(p, a) = r$. By the induction hypothesis, $\hat{\delta}(q, x) = p$. By the definition of $\hat{\delta}$, $\hat{\delta}(q,xa) = \delta(\hat{\delta}(q,x), a) = \delta(p, a) = r$.

-------------------- second one --------------------

This proof shows both directions, since "w contains an odd # 1s → $\hat{\delta}(s, w) = p$" has contrapositive "$\hat{\delta}(s, w) = s$ → w contains an even # 1s"

Base case: $w = \varepsilon$, so that $\hat{\delta}(s, w)=s$. s is not p, and the string has an even number of 1s. ✓

Induction step. Let $w = xa$, and assume by induction that the property is true for the shorter string x.

Case 1: If $\hat{\delta}(s, w)=s$, looking at the state diagram shows us that there are two ways we could have gotten there:

a) $\hat{\delta}(s, x)=s$ and $a=0$. By induction, $\hat{\delta}(s, x)=s$ iff x has an even number of 1s. Since $a=0$, w also has an even number of 1's.

b) $\hat{\delta}(s, x)=p$ and $a=1$. By induction, $\hat{\delta}(s, x)=p$ iff x has an odd number of 1s. Since $a=1$, w has an even number of 1s.

Case 2: If $\hat{\delta}(s, w)=p$, looking at the state diagram shows us that there are two ways we could have gotten there:

a) $\hat{\delta}(s, x)=s$ and $a=1$. By induction, $\hat{\delta}(s, x)=s$ iff x has an even number of 1s. Since $a=1$, w has an even number of 1s.

b) $\hat{\delta}(s, x)=p$ and $a=0$. By induction, $\hat{\delta}(s, x)=p$ iff x has an odd number of 1s. Since $a=0$, w has an odd number of 1s.