MA/CSSE 474
Theory of Computation

Exam Friday; you may bring two sheets of paper.
Material: through HW9.

PDAs and CFGs

Recap: PDA Definition

$M = (K, \Sigma, \Gamma, \Delta, s, A)$, where:

- K is a finite set of states
- Σ is the input alphabet
- Γ is the stack alphabet
- $s \in K$ is the initial state
- $A \subseteq K$ is the set of accepting states, and
- Δ is the transition relation. It is a finite subset of

Σ and Γ are not necessarily disjoint

$$(K \times (\Sigma \cup \{\varepsilon\}) \times \Gamma^*) \times (K \times \Gamma^*)$$

K	$\Sigma \cup \{\varepsilon\}$	Γ^*	K	Γ^*
state	input or ε	string of symbols to pop from top of stack	state	string of symbols to push on top of stack

Recap: Configurations and Yields

A **configuration** of M is an element of $K \times \Sigma^* \times \Gamma^*$.
An **initial configuration** of M is (s, w, ε), where w is the input string.

Let c be any element of $\Sigma \cup \{\varepsilon\}$,
Let γ_1, γ_2 and γ be any elements of Γ^*, and
Let w be any element of Σ^*.
Then:

$(q_1, cw, \gamma_1\gamma) \mid\!-_M (q_2, w, \gamma_2\gamma)$ iff $((q_1, c, \gamma_1), (q_2, \gamma_2)) \in \Delta$.

Let $\mid\!-_M^*$ be the reflexive, transitive closure of $\mid\!-_M$.

C_1 ***yields*** configuration C_2 iff $C_1 \mid\!-_M^* C_2$

Yields

Let c be any element of $\Sigma \cup \{\varepsilon\}$,
Let γ_1, γ_2 and γ be any elements of Γ^*, and
Let w be any element of Σ^*.

Then:
$(q_1, cw, \gamma_1\gamma) \mid\!-_M (q_2, w, \gamma_2\gamma)$ iff $((q_1, c, \gamma_1), (q_2, \gamma_2)) \in \Delta$.

Let $\mid\!-_M^*$ be the reflexive, transitive closure of $\mid\!-_M$.

C_1 ***yields*** configuration C_2 iff $C_1 \mid\!-_M^* C_2$

Recap: Computations and Acceptance

A ***computation*** by M is a finite sequence of configurations C_0, C_1, …, C_n for some $n \geq 0$ such that:

- C_0 is an initial configuration,
- C_n is of the form (q, ε, γ), for some state $q \in K_M$ and some string γ in Γ^*, and
- $C_0 \mid\!\!-_M C_1 \mid\!\!-_M C_2 \mid\!\!-_M \ldots \mid\!\!-_M C_n$.

A computation C of M is an ***accepting computation*** iff:

$C = (s, w, \varepsilon) \mid\!\!-_M^* (q, \varepsilon, \varepsilon)$, and $q \in A$.

M ***accepts*** a string w iff at least one of its computations accepts.

Other paths may:

- Read all the input and halt in a nonaccepting state,
- Read all the input and halt in an accepting state with the stack not empty,
- Loop forever and never finish reading the input, or
- Reach a dead end where no more input can be read.

The ***language accepted by*** M, denoted $L(M)$, is the set of all strings accepted by M.

Rejecting

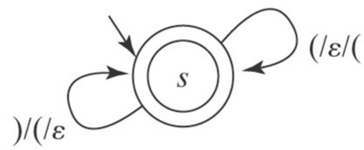
A computation C of M is a ***rejecting computation*** iff:

- $C = (s, w, \varepsilon) \mid\!\!-_M^* (q, w', \alpha)$,
- C is not an accepting computation, and
- M has no moves that it can make from (q, ε, α).

M ***rejects*** a string w iff all of its computations reject.

Note that it is possible that, on input w, M neither accepts nor rejects.

A PDA for Bal

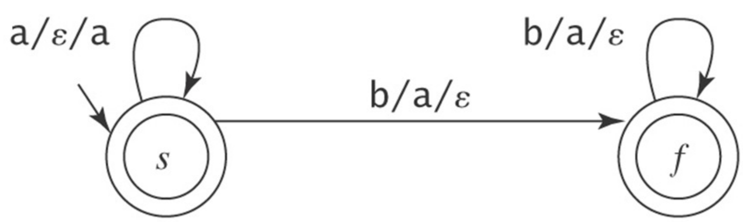


$M = (K, \Sigma, \Gamma, \Delta, s, A)$, where:

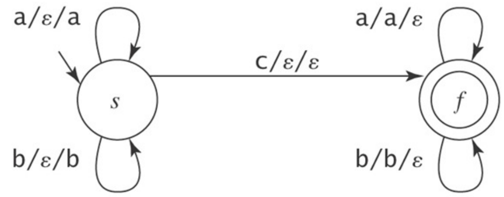
- $K = \{s\}$ the states
- $\Sigma = \{(,)\}$ the input alphabet
- $\Gamma = \{(\}$ the stack alphabet
- $A = \{s\}$
- Δ contains:
  - $((s, (, \varepsilon), (s, ( ))$ **
  - $((s,), ( ), (s, \varepsilon))$

**Important: This does not mean that the stack is empty

A PDA for $A^nB^n = \{a^nb^n : n \geq 0\}$



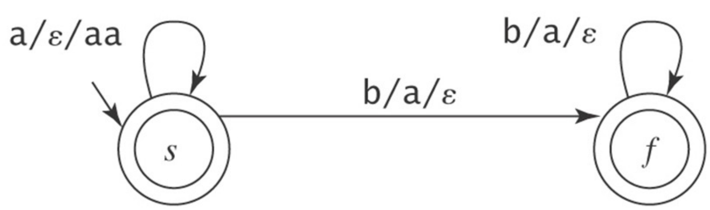
A PDA for $\{wcw^R: w \in \{a, b\}^*\}$



$M = (K, \Sigma, \Gamma, \Delta, s, A)$, where:

- $K = \{s, f\}$ the states
- $\Sigma = \{a, b, c\}$ the input alphabet
- $\Gamma = \{a, b\}$ the stack alphabet
- $A = \{f\}$ the accepting states
- Δ contains: $((s, a, \varepsilon), (s, a))$
  $((s, b, \varepsilon), (s, b))$
  $((s, c, \varepsilon), (f, \varepsilon))$
  $((f, a, a), (f, \varepsilon))$
  $((f, b, b), (f, \varepsilon))$

A PDA for $\{a^n b^{2n}: n \geq 0\}$

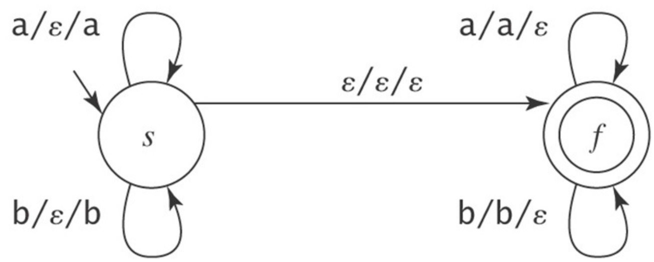


A PDA for PalEven = $\{ww^R : w \in \{a, b\}^*\}$

$S \rightarrow \varepsilon$
$S \rightarrow aSa$
$S \rightarrow bSb$

This one is nondeterministic

A PDA:

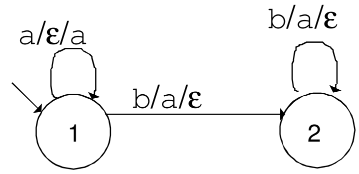


A PDA for $\{w \in \{a, b\}^* : \#_a(w) = \#_b(w)\}$

More on Nondeterminism Accepting Mismatched lengths

$L = \{\mathrm{a}^m\mathrm{b}^n : m \neq n;\ m, n > 0\}$

Start with the case where $n = m$:

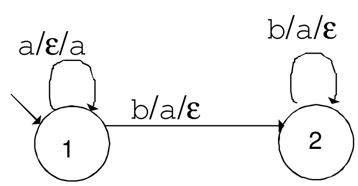


Need to fix it so that

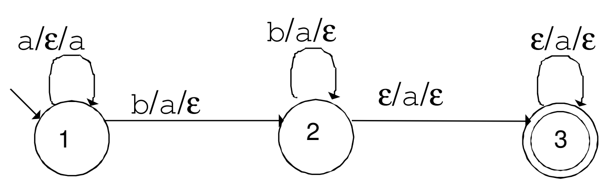
- If stack and input are empty, halt and reject.
- If input is empty but stack is not ($m > n$) (accept):
- If stack is empty but input is not ($m < n$) (accept):

More on Nondeterminism Accepting Mismatches

$L = \{\mathrm{a}^m\mathrm{b}^n : m \neq n;\ m, n > 0\}$

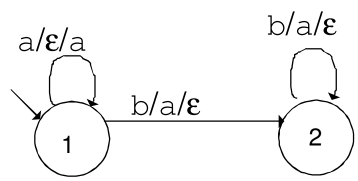


- If input is empty but stack is not ($m < n$) (accept):

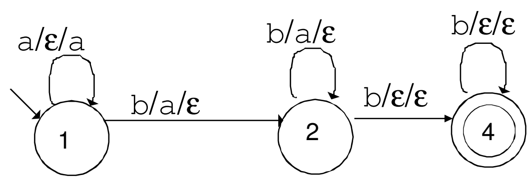


More on Nondeterminism Accepting Mismatches

$L = \{a^m b^n : m \neq n; m, n > 0\}$

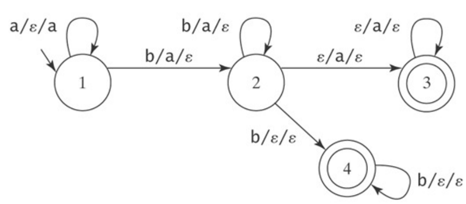


- If stack is empty but input is not ($m > n$) (accept):



Putting It Together

$L = \{a^m b^n : m \neq n; m, n > 0\}$



- Jumping to the input-clearing state 4:
  Need to detect bottom of stack.

- Jumping to the stack-clearing state 3:
  Need to detect end of input.

The Power of Nondeterminism

Consider $A^nB^nC^n = \{a^n b^n c^n : n \geq 0\}$.

PDA for it?

The Power of Nondeterminism

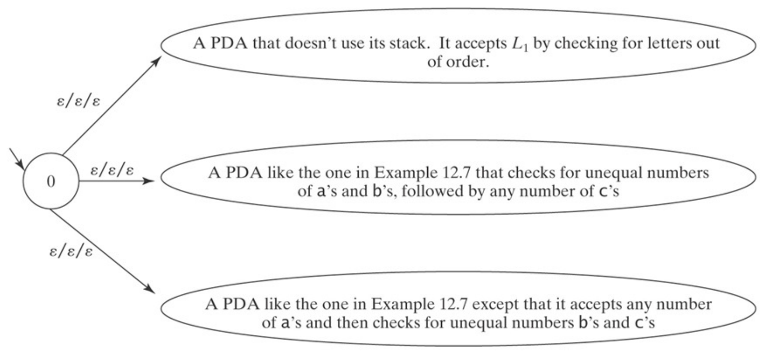
Consider $A^nB^nC^n = \{a^n b^n c^n : n \geq 0\}$.

Now consider $L = \neg A^nB^nC^n$. L is the union of two languages:

1. $\{w \in \{a, b, c\}^*$: the letters are out of order$\}$, and

2. $\{a^i b^j c^k : i, j, k \geq 0$ and $(i \neq j$ or $j \neq k)\}$ (in other words, unequal numbers of a's, b's, and c's).

A PDA for $L = \neg A^nB^nC^n$



Are the Context-Free Languages Closed Under Complement?

$\neg A^nB^nC^n$ is context free.

If the CF languages were closed under complement, then

$$\neg\neg A^nB^nC^n = A^nB^nC^n$$

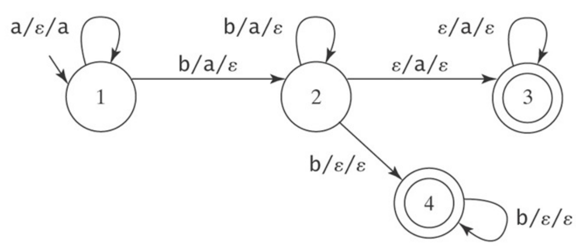
would also be context-free.

But we will prove that it is not.

$L = \{a^n b^m c^p: n, m, p \geq 0$ and $n \neq m$ or $m \neq p\}$

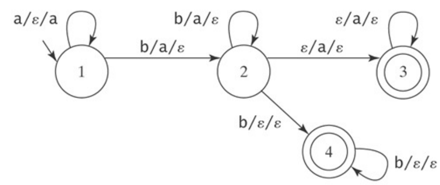
$S \rightarrow NC$	/* $n \neq m$, then arbitrary c's
$S \rightarrow QP$	/* arbitrary a's, then $p \neq m$
$N \rightarrow A$	/* more a's than b's
$N \rightarrow B$	/* more b's than a's
$A \rightarrow \mathtt{a}$	
$A \rightarrow \mathtt{a}A$	
$A \rightarrow \mathtt{a}A\mathtt{b}$	
$B \rightarrow \mathtt{b}$	
$B \rightarrow B\mathtt{b}$	
$B \rightarrow \mathtt{a}B\mathtt{b}$	
$C \rightarrow \varepsilon \mid \mathtt{c}C$	/* add any number of c's
$P \rightarrow B'$	/* more b's than c's
$P \rightarrow C'$	/* more c's than b's
$B' \rightarrow \mathtt{b}$	
$B' \rightarrow \mathtt{b}B'$	
$B' \rightarrow \mathtt{b}B'\mathtt{c}$	
$C' \rightarrow \mathtt{c} \mid C'\mathtt{c}$	
$C' \rightarrow C'\mathtt{c}$	
$C' \rightarrow \mathtt{b}C'\mathtt{c}$	
$Q \rightarrow \varepsilon \mid \mathtt{a}Q$	/* prefix with any number of a's

Reducing Nondeterminism

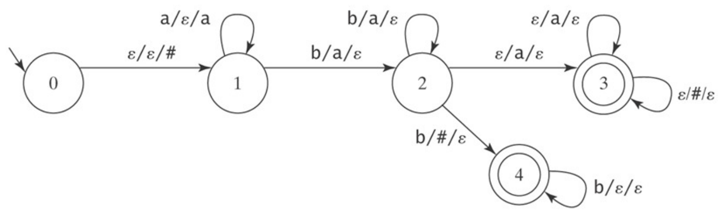


- Jumping to the input-clearing state 4:
  Need to detect bottom of stack, so push # onto the stack before we start.
- Jumping to the stack-clearing state 3:
  Need to detect end of input. Add to *L* a termination character (e.g., $)

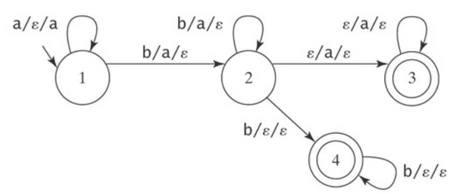
Reducing Nondeterminism



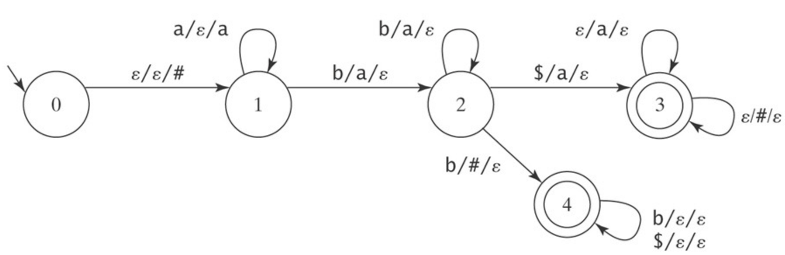
- Jumping to the input-clearing state 4:



Reducing Nondeterminism

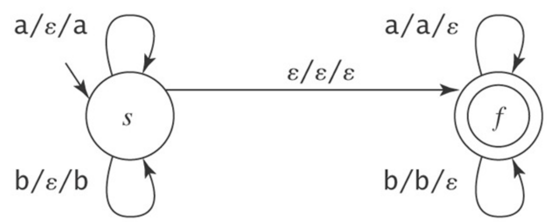


- Jumping to the stack-clearing state 3:



More on PDAs

A PDA for $\{ww^R : w \in \{a, b\}^*\}$:



What about a PDA to accept $\{ww : w \in \{a, b\}^*\}$?

PDAs and Context-Free Grammars

Theorem: The class of languages accepted by PDAs is exactly the class of context-free languages.

Recall: context-free languages are languages that can be defined with context-free grammars.

Restate theorem:

Can describe with context-free grammar

=

Can accept by PDA

Going One Way

Lemma: Each context-free language is accepted by some PDA.

Proof (by construction):

The idea: Let the stack do the work.

Two approaches:

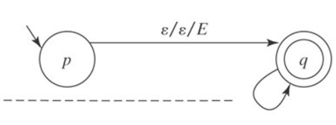
- Top down
- Bottom up

Top Down

The idea: Let the stack keep track of expectations.

Example: Arithmetic expressions

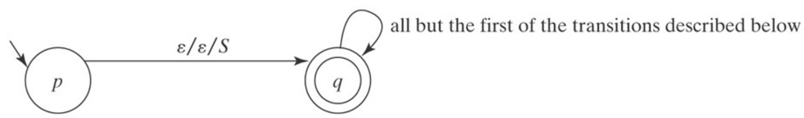
$E \rightarrow E + T$
$E \rightarrow T$
$T \rightarrow T * F$
$T \rightarrow F$
$F \rightarrow (E)$
$F \rightarrow id$



(1) $(q, \varepsilon, E), (q, E+T)$
(2) $(q, \varepsilon, E), (q, T)$
(3) $(q, \varepsilon, T), (q, T*F)$
(4) $(q, \varepsilon, T), (q, F)$
(5) $(q, \varepsilon, F), (q, (E)\,)$
(6) $(q, \varepsilon, F), (q, id)$
(7) $(q, id, id), (q, \varepsilon)$
(8) $(q, (, (\;), (q, \varepsilon)$
(9) $(q,),)\;), (q, \varepsilon)$
(10) $(q, +, +), (q, \varepsilon)$
(11) $(q, *, *), (q, \varepsilon)$

A Top-Down Parser

The outline of *M* is:



$M = (\{p, q\}, \Sigma, V, \Delta, p, \{q\})$, where Δ contains:

- The start-up transition $((p, \varepsilon, \varepsilon), (q, S))$.
- For each rule $X \rightarrow s_1s_2\ldots s_n$. in *R*, the transition:
  $((q, \varepsilon, X), (q, s_1s_2\ldots s_n))$.
- For each character $c \in \Sigma$, the transition:
  $((q, c, c), (q, \varepsilon))$.

Example of the Construction

$L = \{a^n b^* a^n\}$

0 (p, ε, ε), (q, S)

(1) S → ε * 1 (q, ε, S), (q, ε)

(2) S → B 2 (q, ε, S), (q, B)

(3) S → aSa 3 (q, ε, S), (q, aSa)

(4) B → ε 4 (q, ε, B), (q, ε)

(5) B → bB 5 (q, ε, B), (q, bB)

6 (q, a, a), (q, ε)

input = a a b b a a 7 (q, b, b), (q, ε)

trans	***state***	***unread input***	***stack***
	p	a a b b a a	ε
0	q	a a b b a a	S
3	q	a a b b a a	aSa
6	q	a b b a a	Sa
3	q	a b b a a	aSaa
6	q	b b a a	Saa
2	q	b b a a	Baa
5	q	b b a a	bBaa
7	q	b a a	Baa
5	q	b a a	bBaa
7	q	a a	Baa
4	q	a a	aa
6	q	a	a
6	q	ε	ε

Another Example

$L = \{\mathtt{a}^n\mathtt{b}^m\mathtt{c}^p\mathtt{d}^q : m + n = p + q\}$

(1) $S \rightarrow \mathtt{a}S\mathtt{d}$
(2) $S \rightarrow T$
(3) $S \rightarrow U$
(4) $T \rightarrow \mathtt{a}T\mathtt{c}$
(5) $T \rightarrow V$
(6) $U \rightarrow \mathtt{b}U\mathtt{d}$
(7) $U \rightarrow V$
(8) $V \rightarrow \mathtt{b}V\mathtt{c}$
(9) $V \rightarrow \varepsilon$

input = `a a b c d d`

Another Example

$L = \{\mathtt{a}^n\mathtt{b}^m\mathtt{c}^p\mathtt{d}^q : m + n = p + q\}$

(1) $S \rightarrow \mathtt{a}S\mathtt{d}$
(2) $S \rightarrow T$
(3) $S \rightarrow U$
(4) $T \rightarrow \mathtt{a}T\mathtt{c}$
(5) $T \rightarrow V$
(6) $U \rightarrow \mathtt{b}U\mathtt{d}$
(7) $U \rightarrow V$
(8) $V \rightarrow \mathtt{b}V\mathtt{c}$
(9) $V \rightarrow \varepsilon$

input = `a a b c d d`

0 $(p, \varepsilon, \varepsilon), (q, S)$
1 $(q, \varepsilon, S), (q, \mathtt{a}S\mathtt{d})$
2 $(q, \varepsilon, S), (q, T)$
3 $(q, \varepsilon, S), (q, U)$
4 $(q, \varepsilon, T), (q, \mathtt{a}T\mathtt{c})$
5 $(q, \varepsilon, T), (q, V)$
6 $(q, \varepsilon, U), (q, \mathtt{b}U\mathtt{d})$
7 $(q, \varepsilon, U), (q, V)$
8 $(q, \varepsilon, V), (q, \mathtt{b}V\mathtt{c})$
9 $(q, \varepsilon, V), (q, \varepsilon)$
10 $(q, \mathtt{a}, \mathtt{a}), (q, \varepsilon)$
11 $(q, \mathtt{b}, \mathtt{b}), (q, \varepsilon)$
12 $(q, \mathtt{c}, \mathtt{c}), (q, \varepsilon)$
13 $(q, \mathtt{d}, \mathtt{d}), (q, \varepsilon)$

trans	***state***	***unread input***	***stack***

Notice Nondeterminism

Machines constructed with the algorithm are often nondeterministic, even when they needn't be. This happens even with trivial languages.

Example: $A^nB^n = \{a^nb^n : n \geq 0\}$

A grammar for A^nB^n is:

[1] $S \rightarrow aSb$
[2] $S \rightarrow \varepsilon$

A PDA M for A^nB^n is:

(0) $((p, \varepsilon, \varepsilon), (q, S))$
(1) $((q, \varepsilon, S), (q, aSb))$
(2) $((q, \varepsilon, S), (q, \varepsilon))$
(3) $((q, a, a), (q, \varepsilon))$
(4) $((q, b, b), (q, \varepsilon))$

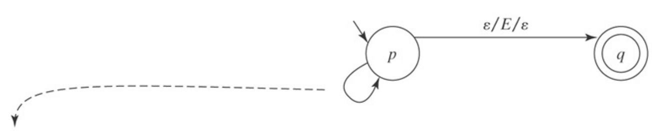
But transitions 1 and 2 make M nondeterministic.

A directly constructed machine for A^nB^n:

Bottom-Up

The idea: Let the stack keep track of what has been found.

(1) $E \rightarrow E + T$
(2) $E \rightarrow T$
(3) $T \rightarrow T * F$
(4) $T \rightarrow F$
(5) $F \rightarrow (E)$
(6) $F \rightarrow id$

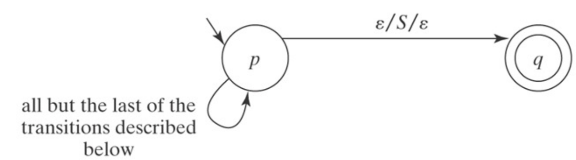


Reduce Transitions:
(1) $(p, \varepsilon, T + E), (p, E)$
(2) $(p, \varepsilon, T), (p, E)$
(3) $(p, \varepsilon, F * T), (p, T)$
(4) $(p, \varepsilon, F), (p, T)$
(5) $(p, \varepsilon,)E( ), (p, F)$
(6) $(p, \varepsilon, \text{id}), (p, F)$

Shift Transitions
(7) $(p, \text{id}, \varepsilon), (p, \text{id})$
(8) $(p, (, \varepsilon), (p, ()$
(9) $(p,), \varepsilon), (p,))$
(10) $(p, +, \varepsilon), (p, +)$
(11) $(p, *, \varepsilon), (p, *)$

A Bottom-Up Parser

The outline of *M* is:



$M = (\{p, q\}, \Sigma, V, \Delta, p, \{q\})$, where Δ contains:

- The shift transitions: $((p, c, \varepsilon), (p, c))$, for each $c \in \Sigma$.
- The reduce transitions: $((p, \varepsilon, (s_1 s_2 \ldots s_n.)^R), (p, X))$, for each rule $X \rightarrow s_1 s_2 \ldots s_n.$ in *G*.
- The finish up transition: $((p, \varepsilon, S), (q, \varepsilon))$.