MA/CSSE 474
Theory of Computation

Removing Ambiguity
Chomsky Normal Form
Pushdown Automata

Recap: Ambiguity

A grammar is ***ambiguous*** iff there is at least one string in *L*(*G*) for which *G* produces more than one parse tree.

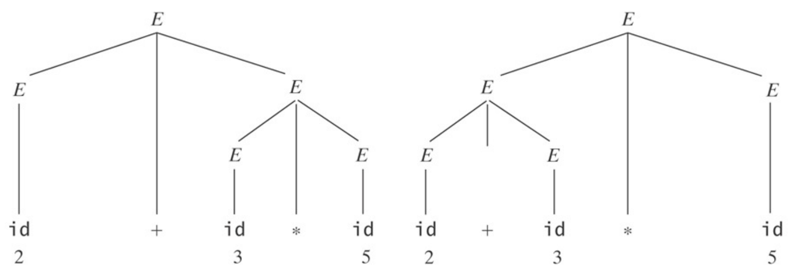
For many applications of context-free grammars, this is a problem.

Example: A programming language.

- If there can be two different structures for a string in the language, there can be two different meanings.
- Not good!

An Arithmetic Expression Grammar

$E \rightarrow E + E$
$E \rightarrow E * E$
$E \rightarrow (E)$
$E \rightarrow \texttt{id}$



Inherent Ambiguity

Some CF languages have the property that every grammar for them is ambiguous. We call such languages ***inherently ambiguous***.

Example:

$L = \{a^n b^n c^m : n, m \geq 0\} \cup \{a^n b^m c^m : n, m \geq 0\}$.

Inherent Ambiguity

$L = \{a^n b^n c^m : n, m \geq 0\} \cup \{a^n b^m c^m : n, m \geq 0\}$.

One grammar for L has the rules:

$S \rightarrow S_1 \mid S_2$

$S_1 \rightarrow S_1 c \mid A$ /* Generate all strings in $\{a^n b^n c^m\}$.
$A \rightarrow aAb \mid \varepsilon$

$S_2 \rightarrow aS_2 \mid B$ /* Generate all strings in $\{a^n b^m c^m\}$.
$B \rightarrow bBc \mid \varepsilon$

Consider any string of the form $a^n b^n c^n$.

It turns out that L is inherently ambiguous.

Inherent Ambiguity

Both of the following problems are undecidable:

- Given a context-free grammar G, is G ambiguous?
- Given a context-free language L, is L inherently ambiguous?

But We Can Often Reduce Ambiguity

We can get rid of:

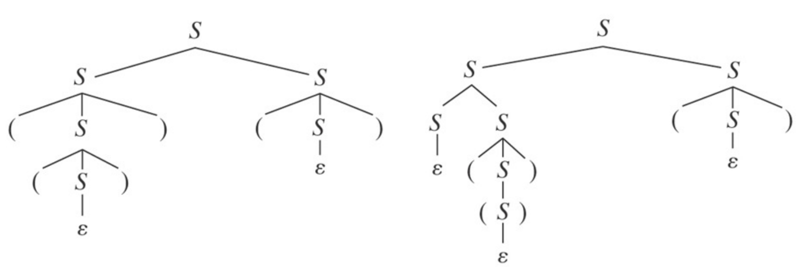
- some ε rules like $S \to \varepsilon$,
- rules with symmetric right-hand sides, e.g.,

  $S \to SS$
  $E \to E + E$

- rule sets that lead to ambiguous attachment of optional postfixes.

A Highly Ambiguous Grammar

$S \to \varepsilon$
$S \to SS$
$S \to (S)$



Resolving the Ambiguity with a Different Grammar

The biggest problem is the ε rule.

A different grammar for the language of balanced parentheses:

$S^* \rightarrow \varepsilon$
$S^* \rightarrow S$
$S \rightarrow SS$
$S \rightarrow (S)$
$S \rightarrow ()$

We'd like to have an algorithm for removing all ε-productions…
… except for the case where ε- is actually in the language;
then we introduce a new start symbol and have one ε-production whose left side is that symbol.

Nullable Nonterminals

Examples:

$S \rightarrow \mathrm{a}T\mathrm{a}$
$T \rightarrow \varepsilon$

$S \rightarrow \mathrm{a}T\mathrm{a}$
$T \rightarrow A\ B$
$A \rightarrow \varepsilon$
$B \rightarrow \varepsilon$

A nonterminal X is ***nullable*** iff either:
(1) there is a rule $X \rightarrow \varepsilon$, or
(2) there is a rule $X \rightarrow PQR\ldots$ and P, Q, R, … are all nullable.

Nullable Nonterminals

A nonterminal X is ***nullable*** iff either:
(1) there is a rule $X \rightarrow \varepsilon$, or
(2) there is a rule $X \rightarrow PQR\ldots$ and P, Q, R, … are all nullable.

So compute N, the set of nullable nonterminals, as follows:

1. Set N to the set of nonterminals that satisfy (1).
2. Repeat until an entire pass is made without adding anything to N
   Evaluate all other nonterminals with respect to (2).
   If any nonterminal satisfies (2) and is not in N, insert it.

A General Technique for Getting Rid of ε-Rules

Definition: a rule is ***modifiable*** iff it is of the form:

$P \rightarrow \alpha Q \beta$, for some nullable Q.

removeEps(G: cfg) =
1. Let $G' = G$.
2. Find the set N of nullable nonterminals in G'.
3. Repeat until G' contains no modifiable rules that haven't been processed:
   Given the rule $P \rightarrow \alpha Q \beta$, where $Q \in N$,
   add the rule $P \rightarrow \alpha\beta$
   if it is not already present and if $\alpha\beta \neq \varepsilon$ and if $P \neq \alpha\beta$.
4. Delete from G' all rules of the form $X \rightarrow \varepsilon$.
5. Return G'.

$L(G') = L(G) - \{\varepsilon\}$

An Example

$G = \{\{S, T, A, B, C, \texttt{a}, \texttt{b}, \texttt{c}\}, \{\texttt{a}, \texttt{b}, \texttt{c}\}, R, S\}$,

$R = \{ S \rightarrow \texttt{a}T\texttt{a}$
$T \rightarrow ABC$
$A \rightarrow \texttt{a}A \mid C$
$B \rightarrow B\texttt{b} \mid C$
$C \rightarrow \texttt{c} \mid \varepsilon \}$

removeEps(G: cfg) =
1. Let $G' = G$.
2. Find the set N of nullable nonterminals in G'.
3. Repeat until G' contains no modifiable rules that haven't been processed:
   Given the rule $P \rightarrow \alpha Q \beta$, where $Q \in N$, add the rule $P \rightarrow \alpha\beta$
   if it is not already present and if $\alpha\beta \neq \varepsilon$ and if $P \neq \alpha\beta$.
4. Delete from G' all rules of the form $X \rightarrow \varepsilon$.
5. Return G'.

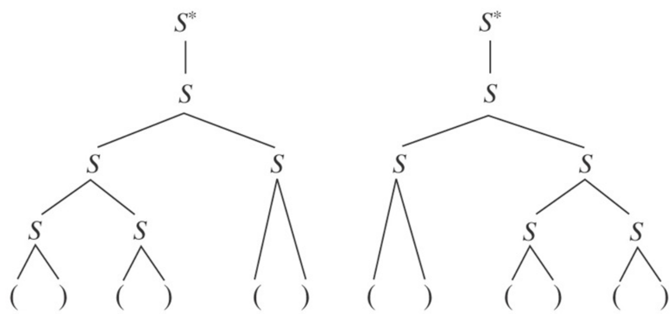
What If $\varepsilon \in L$?

atmostoneEps(G: cfg) =
1. G'' = *removeEps*(G).
2. If S_G is nullable then /* i. e., $\varepsilon \in L(G)$
   2.1 Create in G'' a new start symbol S^*.
   2.2 Add to $R_{G''}$ the two rules:
   $S^* \rightarrow \varepsilon$
   $S^* \rightarrow S_G$.
3. Return G''.

But There is Still Ambiguity

$S^* \rightarrow \varepsilon$
$S^* \rightarrow S$
$S \rightarrow SS$
$S \rightarrow (S)$
$S \rightarrow ()$

What about ()()() ?



Eliminating Symmetric Recursive Rules

$S^* \rightarrow \varepsilon$
$S^* \rightarrow S$
$S \rightarrow SS$
$S \rightarrow (S)$
$S \rightarrow ()$

Replace $S \rightarrow SS$ with one of:

$S \rightarrow SS_1$ /* force branching to the left
$S \rightarrow S_1S$ /* force branching to the right

So we get:

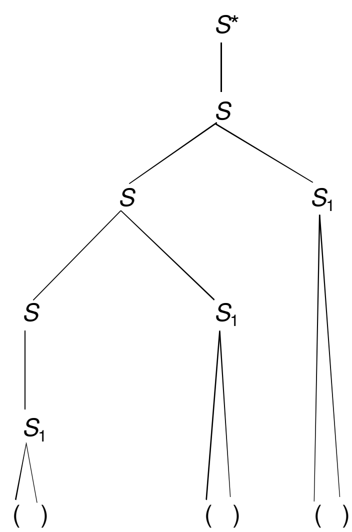
$S^* \rightarrow \varepsilon$
$S^* \rightarrow S$

$S \rightarrow SS_1$
$S \rightarrow S_1$
$S_1 \rightarrow (S)$
$S_1 \rightarrow ()$

Eliminating Symmetric Recursive Rules

So we get:

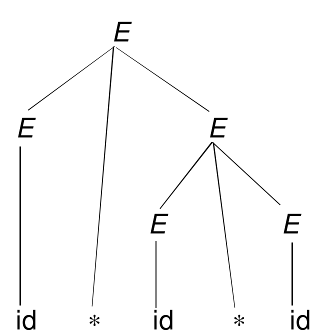
$S^* \rightarrow \varepsilon$
$S^* \rightarrow S$
$S \rightarrow SS_1$
$S \rightarrow S_1$
$S_1 \rightarrow (S)$
$S_1 \rightarrow ()$

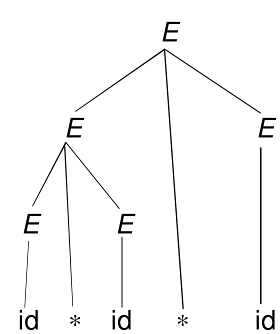


Arithmetic Expressions

$E \rightarrow E + E$
$E \rightarrow E * E$
$E \rightarrow (E)$
$E \rightarrow$ `id` }

Problem 1: Associativity

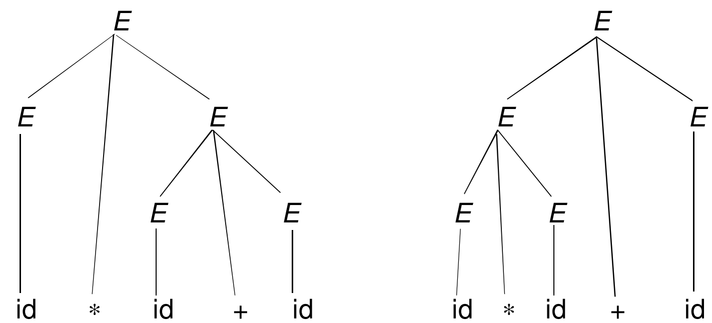




Arithmetic Expressions

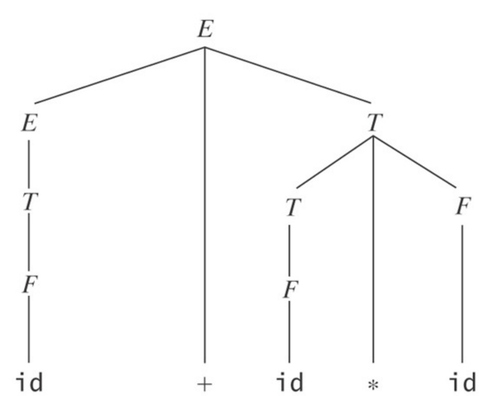
$E \rightarrow E + E$
$E \rightarrow E * E$
$E \rightarrow (E)$
$E \rightarrow \texttt{id}$ }

Problem 2: Precedence



Arithmetic Expressions - A Better Way

$E \rightarrow E + T$
$E \rightarrow T$
$T \rightarrow T * F$
$T \rightarrow F$
$F \rightarrow (E)$
$F \rightarrow \text{id}$



Ambiguous Attachment

The dangling else problem:

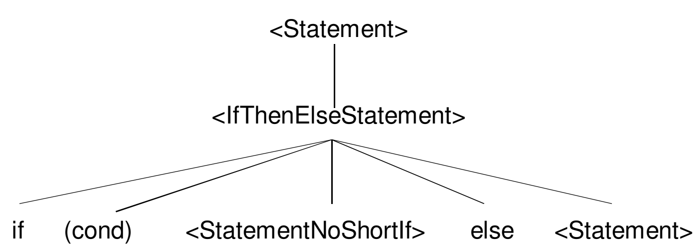
<stmt> ::= `if` <cond> `then` <stmt>
<stmt> ::= `if` <cond> `then` <stmt> `else` <stmt>

Consider:

`if` $cond_1$ `then` `if` $cond_2$ `then` st_1 `else` st_2

The Java Fix

<Statement> ::= <IfThenStatement> | <IfThenElseStatement> |
<IfThenElseStatementNoShortIf>
<StatementNoShortIf> ::= <block> |
<IfThenElseStatementNoShortIf> | …
<IfThenStatement> ::= `if` (<Expression>) <Statement>
<IfThenElseStatement> ::= `if` (<Expression>)
<StatementNoShortIf> `else` <Statement>
<IfThenElseStatementNoShortIf> ::=
`if` (<Expression>) <StatementNoShortIf>
`else` <StatementNoShortIf>



Going Too Far

S → *NP VP*
NP → the *Nominal* | *Nominal* | *ProperNoun* | *NP PP*
Nominal → *N* | *Adjs N*
N → `cat` | `girl` | `dogs` | `ball` | `chocolate` | `bat`
ProperNoun → `Chris` | `Fluffy`
Adjs → *Adj Adjs* | *Adj*
Adj → `young` | `older` | `smart`
VP → *V* | *V NP* | *VP PP*
V → `like` | `likes` | `thinks` | `hits`
PP → *Prep NP*
Prep → `with`

- `Chris likes the girl with the cat.`
- `Chris shot the bear with a rifle.`

Going Too Far

- `Chris likes the girl with the cat.`
- `Chris shot the bear with a rifle.`

- `Chris shot the bear with a rifle.`

Comparing Regular and Context-Free Languages

Regular Languages	Context-Free Languages
• regular exprs. or	
• regular grammars	• context-free grammars
• recognize	• parse

Normal Forms

A normal form *F* for a set *C* of data objects is a form, i.e., a set of syntactically valid objects, with the following two properties:

- For every element *c* of *C*, except possibly a finite set of special cases, there exists some element *f* of *F* such that *f* is equivalent to *c* with respect to some set of tasks.

- *F* is simpler than the original form in which the elements of *C* are written. By “simpler” we mean that at least some tasks are easier to perform on elements of *F* than they would be on elements of *C*.

Normal Forms

If you want to design algorithms, it is often useful to have a limited number of input forms that you have to deal with.

Normal forms are designed to do just that. Various ones have been developed for various purposes.

Examples:

- Disjunctive normal form for database queries so that they can be entered in a query-by-example grid.

- Jordan normal form for a square matrix, in which the matrix is almost diagonal in the sense that its only non-zero entries lie on the diagonal and the superdiagonal.

- Various normal forms for grammars to support specific parsing techniques.

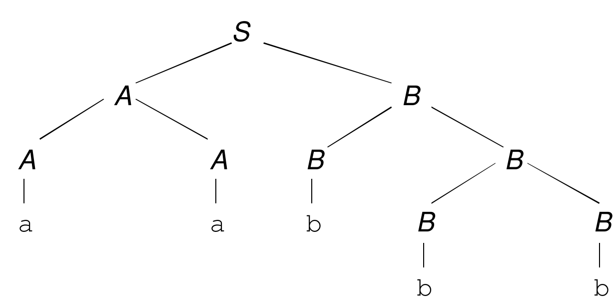
Normal Forms for Grammars

Chomsky Normal Form, in which all rules are of one of the following two forms:

- $X \rightarrow a$, where $a \in \Sigma$, or
- $X \rightarrow BC$, where B and C are elements of $V - \Sigma$.

Advantages:

- Parsers can use binary trees.
- Exact length of derivations is known:



Normal Forms for Grammars

Greibach Normal Form, in which all rules are of the following form:

- $X \rightarrow a\,\beta$, where $a \in \Sigma$ and $\beta \in (V - \Sigma)^*$.

Advantages:

- Every derivation of a string *s* contains |*s*| rule applications.
- Greibach normal form grammars can easily be converted to pushdown automata with no ε-transitions. This is useful because such PDAs are guaranteed to halt.

Normal Forms Exist

Theorem: Given a CFG *G*, there exists an equivalent Chomsky normal form grammar G_C such that:

$$L(G_C) = L(G) - \{\varepsilon\}.$$

Proof: The proof is by construction.

Details of both are complex but straightforward; I leave them for you to read in the textbook and/or in the next 16 slides.

Theorem: Given a CFG *G*, there exists an equivalent Greibach normal form grammar G_G such that:

$$L(G_G) = L(G) - \{\varepsilon\}.$$

Proof: The proof is also by construction.

Converting to a Normal Form

1. Apply some transformation to G to get rid of undesirable property 1. Show that the language generated by G is unchanged.

2. Apply another transformation to G to get rid of undesirable property 2. Show that the language generated by G is unchanged *and* that undesirable property 1 has not been reintroduced.

3. Continue until the grammar is in the desired form.

Rule Substitution

$X \rightarrow \mathtt{a}Y\mathtt{c}$
$Y \rightarrow \mathtt{b}$
$Y \rightarrow ZZ$

We can replace the X rule with the rules:

$X \rightarrow \mathtt{abc}$
$X \rightarrow \mathtt{a}ZZ\mathtt{c}$

$X \Rightarrow \mathtt{a}Y\mathtt{c} \Rightarrow \mathtt{a}ZZ\mathtt{c}$

Rule Substitution

Theorem: Let G contain the rules:

$$X \to \alpha Y\beta \quad \text{and} \quad Y \to \gamma_1 \mid \gamma_2 \mid \ldots \mid \gamma_n,$$

Replace $X \to \alpha Y\beta$ by:

$$X \to \alpha\gamma_1\beta, \quad X \to \alpha\gamma_2\beta, \quad \ldots, \quad X \to \alpha\gamma_n\beta.$$

The new grammar G' will be equivalent to G.

Rule Substitution

Replace $X \to \alpha Y\beta$ by:
$X \to \alpha\gamma_1\beta, \quad X \to \alpha\gamma_2\beta, \quad \ldots, X \to \alpha\gamma_n\beta.$

Proof:

● Every string in $L(G)$ is also in $L(G')$:

If $X \to \alpha Y\beta$ is not used, then use same derivation.
If it is used, then one derivation is:
$S \Rightarrow \ldots \Rightarrow \delta X\phi \Rightarrow \delta\alpha Y\beta\phi \Rightarrow \delta\alpha\gamma_k\beta\phi \Rightarrow \ldots \Rightarrow w$

Use this one instead:
$S \Rightarrow \ldots \Rightarrow \delta X\phi \Rightarrow \quad \delta\alpha\gamma_k\beta\phi \Rightarrow \ldots \Rightarrow w$

● Every string in $L(G')$ is also in $L(G)$: Every new rule can be simulated by old rules.

Conversion to Chomsky Normal Form

1. Remove all ε-rules, using the algorithm *removeEps*.
2. Remove all unit productions (rules of the form $A \rightarrow B$).
3. Remove all rules whose right hand sides have length greater than 1 and include a terminal:

   (e.g., $A \rightarrow \mathrm{a}B$ or $A \rightarrow B\mathrm{a}C$)
4. Remove all rules whose right hand sides have length greater than 2:

   (e.g., $A \rightarrow BCDE$)

Recap: Removing ε-Productions

Remove all ε productions:

(1) If there is a rule $P \rightarrow \alpha Q \beta$ and Q is nullable,

Then: Add the rule $P \rightarrow \alpha\beta$.

(2) Delete all rules $Q \rightarrow \varepsilon$.

Removing ε-Productions

Example:

$S \rightarrow \mathtt{a}A$
$A \rightarrow B \mid CDC$
$B \rightarrow \varepsilon$
$B \rightarrow \mathtt{a}$
$C \rightarrow BD$
$D \rightarrow \mathtt{b}$
$D \rightarrow \varepsilon$

Unit Productions

A ***unit production*** is a rule whose right-hand side consists of a single nonterminal symbol.

Example:

$S \rightarrow X\ Y$
$X \rightarrow A$
$A \rightarrow B \mid \mathtt{a}$
$B \rightarrow \mathtt{b}$
$Y \rightarrow T$
$T \rightarrow Y \mid \mathtt{c}$

Removing Unit Productions

removeUnits(*G*) =

1. Let *G′* = *G*.
2. Until no unit productions remain in *G′* do:
   - 2.1 Choose some unit production $X \rightarrow Y$.
   - 2.2 Remove it from *G′*.
   - 2.3 Consider only rules that still remain. For every rule $Y \rightarrow \beta$, where $\beta \in V^*$, do:
     - Add to *G′* the rule $X \rightarrow \beta$ unless it is a rule that has already been removed once.
3. Return *G′*.

After removing epsilon productions and unit productions, all rules whose right hand sides have length 1 are in Chomsky Normal Form.

Removing Unit Productions

removeUnits(*G*) =

1. Let *G′* = *G*.
2. Until no unit productions remain in *G′* do:
   - 2.1 Choose some unit production $X \rightarrow Y$.
   - 2.2 Remove it from *G′*.
   - 2.3 Consider only rules that still remain. For every rule $Y \rightarrow \beta$, where $\beta \in V^*$, do:
     - Add to *G′* the rule $X \rightarrow \beta$ unless it is a rule that has already been removed once.
3. Return *G′*.

Example:

$S \rightarrow X\ Y$
$X \rightarrow A$
$A \rightarrow B \mid$ `a`
$B \rightarrow$ `b`
$Y \rightarrow T$
$T \rightarrow Y \mid$ `c`

Mixed Rules

removeMixed(*G*) =

1. Let $G' = G$.
2. Create a new nonterminal T_a for each terminal a in Σ.
3. Modify each rule whose right-hand side has length greater than 1 and that contains a terminal symbol by substituting T_a for each occurrence of the terminal a.
4. Add to G, for each T_a, the rule $T_a \rightarrow a$.
5. Return G'.

Example:

$A \rightarrow \mathrm{a}$
$A \rightarrow \mathrm{a}\ B$
$A \rightarrow B\mathrm{a}C$
$A \rightarrow B\mathrm{b}C$

Long Rules

removeLong(*G*) =

1. Let $G' = G$.
2. For each rule r of the form:

$$A \rightarrow N_1 N_2 N_3 N_4 \ldots N_n,\ n > 2$$

create new nonterminals $M_2, M_3, \ldots M_{n-1}$.

3. Replace r with the rule $A \rightarrow N_1 M_2$.
4. Add the rules:

$M_2 \rightarrow N_2 M_3$,
$M_3 \rightarrow N_3 M_4$, …
$M_{n-1} \rightarrow N_{n-1} N_n$.

5. Return G'.

Example:
$A \rightarrow BCDEF$

An Example

$S \rightarrow aACa$
$A \rightarrow B \mid a$
$B \rightarrow C \mid c$
$C \rightarrow cC \mid \varepsilon$

removeEps returns:

$S \rightarrow aACa \mid aAa \mid aCa \mid aa$
$A \rightarrow B \mid a$
$B \rightarrow C \mid c$
$C \rightarrow cC \mid c$

An Example

$S \rightarrow aACa \mid aAa \mid aCa \mid aa$
$A \rightarrow B \mid a$
$B \rightarrow C \mid c$
$C \rightarrow cC \mid c$

Next we apply *removeUnits*:
Remove $A \rightarrow B$. Add $A \rightarrow C \mid c$.
Remove $B \rightarrow C$. Add $B \rightarrow cC$ ($B \rightarrow c$, already there).
Remove $A \rightarrow C$. Add $A \rightarrow cC$ ($A \rightarrow c$, already there).

So *removeUnits* returns:

$S \rightarrow aACa \mid aAa \mid aCa \mid aa$
$A \rightarrow a \mid c \mid cC$
$B \rightarrow c \mid cC$
$C \rightarrow cC \mid c$

An Example

$S \rightarrow \mathtt{a}AC\mathtt{a} \mid \mathtt{a}A\mathtt{a} \mid \mathtt{a}C\mathtt{a} \mid \mathtt{aa}$
$A \rightarrow \mathtt{a} \mid \mathtt{c} \mid \mathtt{c}C$
$B \rightarrow \mathtt{c} \mid \mathtt{c}C$
$C \rightarrow \mathtt{c}C \mid \mathtt{c}$

Next we apply *removeMixed*, which returns:

$S \rightarrow T_aACT_a \mid T_aAT_a \mid T_aCT_a \mid T_aT_a$
$A \rightarrow \mathtt{a} \mid \mathtt{c} \mid T_cC$
$B \rightarrow \mathtt{c} \mid T_cC$
$C \rightarrow T_cC \mid \mathtt{c}$
$T_a \rightarrow \mathtt{a}$
$T_c \rightarrow \mathtt{c}$

An Example

$S \rightarrow T_aACT_a \mid T_aAT_a \mid T_aCT_a \mid T_aT_a$
$A \rightarrow \mathtt{a} \mid \mathtt{c} \mid T_cC$
$B \rightarrow \mathtt{c} \mid T_cC$
$C \rightarrow T_cC \mid \mathtt{c}$
$T_a \rightarrow \mathtt{a}$
$T_c \rightarrow \mathtt{c}$

Finally, we apply *removeLong*, which returns:

$S \rightarrow T_aS_1$ $\quad S \rightarrow T_aS_3$ $\quad S \rightarrow T_aS_4$ $\quad S \rightarrow T_aT_a$
$S_1 \rightarrow AS_2$ $\quad S_3 \rightarrow AT_a$ $\quad S_4 \rightarrow CT_a$
$S_2 \rightarrow CT_a$
$A \rightarrow \mathtt{a} \mid \mathtt{c} \mid T_cC$
$B \rightarrow \mathtt{c} \mid T_cC$
$C \rightarrow T_cC \mid \mathtt{c}$
$T_a \rightarrow \mathtt{a}$
$T_c \rightarrow \mathtt{c}$

The Price of Normal Forms

$E \to E + E$
$E \to (E)$
$E \to \mathtt{id}$

Converting to Chomsky normal form:

$E \to E\ E'$
$E' \to P\ E$
$E \to L\ E''$
$E'' \to E\ R$
$E \to \mathtt{id}$
$L \to ($
$R \to)$
$P \to +$

Conversion doesn't change weak generative capacity but it may change strong generative capacity.

Pushdown Automata

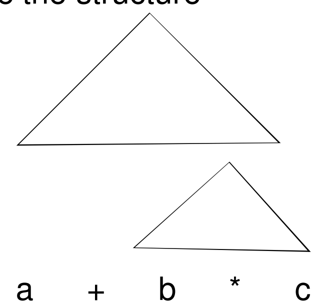
Recognizing Context-Free Languages

Two notions of recognition:

(1) Say yes or no, just like with FSMs

(2) Say yes or no, AND

if yes, describe the structure



Definition of a Pushdown Automaton

$M = (K, \Sigma, \Gamma, \Delta, s, A)$, where:

- K is a finite set of states
- Σ is the input alphabet
- Γ is the stack alphabet
- $s \in K$ is the initial state
- $A \subseteq K$ is the set of accepting states, and
- Δ is the transition relation. It is a finite subset of

Σ and Γ are not necessarily disjoint

$$(K \times (\Sigma \cup \{\varepsilon\}) \times \Gamma^*) \times (K \times \Gamma^*)$$

K	$\Sigma \cup \{\varepsilon\}$	Γ^*	K	Γ^*
state	input or ε	string of symbols to pop from top of stack	state	string of symbols to push on top of stack

Definition of a Pushdown Automaton

A **configuration** of M is an element of $K \times \Sigma^* \times \Gamma^*$.

An **initial configuration** of M is (s, w, ε), where w is the input string.

Manipulating the Stack

c
a
b

will be written as `cab`

If $c_1c_2\ldots c_n$ is pushed onto the stack:

c_1
c_2
c_n
c
a
b

$c_1c_2\ldots c_n$`cab`

Yields

Let c be any element of $\Sigma \cup \{\varepsilon\}$,
Let γ_1, γ_2 and γ be any elements of Γ^*, and
Let w be any element of Σ^*.

Then:
$(q_1, cw, \gamma_1\gamma) \mid\!-_M (q_2, w, \gamma_2\gamma)$ iff $((q_1, c, \gamma_1), (q_2, \gamma_2)) \in \Delta$.

Let $\mid\!-_M{}^*$ be the reflexive, transitive closure of $\mid\!-_M$.

C_1 ***yields*** configuration C_2 iff C_1 **$\mid\!-_M{}^*$** C_2

Computations

A ***computation*** by M is a finite sequence of configurations $C_0, C_1, \ldots, C_n$ for some $n \geq 0$ such that:

- C_0 is an initial configuration,
- C_n is of the form (q, ε, γ), for some state $q \in K_M$ and some string γ in Γ^*, and
- $C_0 \mid\!-_M C_1 \mid\!-_M C_2 \mid\!-_M \ldots \mid\!-_M C_n$.

Nondeterminism

If M is in some configuration (q_1, s, γ) it is possible that:

- Δ contains exactly one transition that matches.
- Δ contains more than one transition that matches.
- Δ contains no transition that matches.

Accepting

A computation C of M is an ***accepting computation*** iff:

- $C = (s, w, \varepsilon) \mid\!-_M^* (q, \varepsilon, \varepsilon)$, and
- $q \in A$.

M ***accepts*** a string w iff at least one of its computations accepts.

Other paths may:

- Read all the input and halt in a nonaccepting state,
- Read all the input and halt in an accepting state with the stack not empty,
- Loop forever and never finish reading the input, or
- Reach a dead end where no more input can be read.

The ***language accepted by*** M, denoted $L(M)$, is the set of all strings accepted by M.

Rejecting

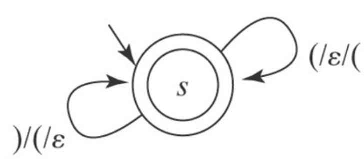
A computation *C* of *M* is a ***rejecting computation*** iff:

- $C = (s, w, \varepsilon) \vdash_M^* (q, w', \alpha)$,
- *C* is not an accepting computation, and
- *M* has no moves that it can make from (q, ε, α).

M ***rejects*** a string *w* iff all of its computations reject.

Note that it is possible that, on input *w*, *M* neither accepts nor rejects.

A PDA for Bal



$M = (K, \Sigma, \Gamma, \Delta, s, A)$, where:

$K = \{s\}$	the states
$\Sigma = \{(,)\}$	the input alphabet
$\Gamma = \{(\}$	the stack alphabet
$A = \{s\}$	

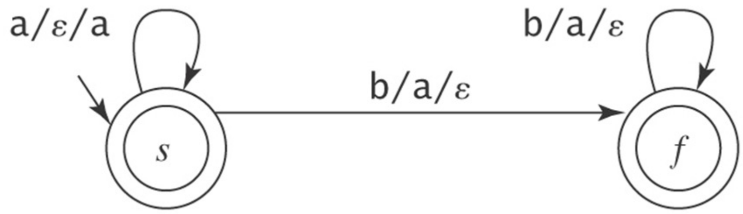
Δ contains:

$((s, (, \varepsilon), (s, (\,))$ **

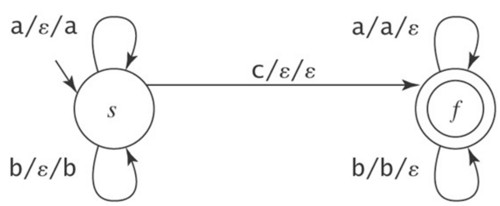
$((s,), (\,), (s, \varepsilon))$

**Important: This does not mean that the stack is empty

A PDA for $A^nB^n = \{a^nb^n: n \geq 0\}$



A PDA for $\{wcw^R: w \in \{a, b\}^*\}$

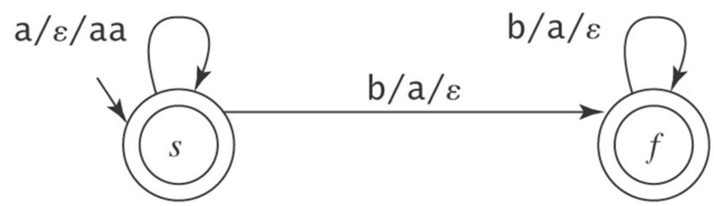


$M = (K, \Sigma, \Gamma, \Delta, s, A)$, where:

$K = \{s, f\}$	the states
$\Sigma = \{a, b, c\}$	the input alphabet
$\Gamma = \{a, b\}$	the stack alphabet
$A = \{f\}$	the accepting states

Δ contains:
- $((s, a, \varepsilon), (s, a))$
- $((s, b, \varepsilon), (s, b))$
- $((s, c, \varepsilon), (f, \varepsilon))$
- $((f, a, a), (f, \varepsilon))$
- $((f, b, b), (f, \varepsilon))$

A PDA for $\{a^n b^{2n}: n \geq 0\}$

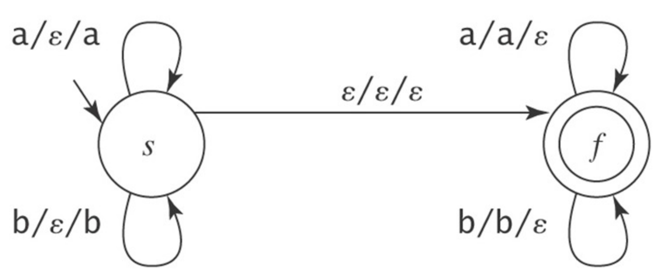


A PDA for PalEven $=\{ww^R: w \in \{a, b\}^*\}$

$S \rightarrow \varepsilon$
$S \rightarrow aSa$
$S \rightarrow bSb$

This one is nondeterministic

A PDA:



A PDA for $\{w \in \{\texttt{a}, \texttt{b}\}^* : \#_a(w) = \#_b(w)\}$