Former 474 HW 1 problems, now optional HW0
(highlighted problems were the ones to turn in)

Note on problem A.5:

In line 3, it says, "Define the following predicates over those sets."

A clearer statement would be "Assume that the following predicates over those sets have been defined."

That part of the problem is not asking you to do anything; instead it is giving you the building blocks to use for what you are supposed to do, which is parts a and b.

1. Prove each of the following:

a. $((A \wedge B) \rightarrow C) \leftrightarrow (\neg A \vee \neg B \vee C)$.

b. $(A \wedge \neg B \wedge \neg C) \rightarrow (A \vee \neg(B \wedge C))$.

2. List the elements of each of the following sets:

a. $\mathcal{P}(\{\text{apple}, \text{pear}, \text{banana}\})$.

b. $\mathcal{P}(\{a, b\}) - \mathcal{P}(\{a, c\})$.

c. $\mathcal{P}(\varnothing)$.

d. $\{a, b\} \times \{1, 2, 3\} \times \varnothing$.

e. $\{x \in \mathbb{N}: (x \leq 7 \wedge x \geq 7)\}$.

f. $\{x \in \mathbb{N}: \exists y \in \mathbb{N}\,(y < 10 \wedge (y + 2 = x))\}$ (where $\mathbb{N}$ is the set of nonnegative integers).

g. $\{x \in \mathbb{N}: \exists y \in \mathbb{N}\,(\exists z \in \mathbb{N}\,((x = y + z) \wedge (y < 5) \wedge (z < 4)))\}$.

A.1 A.2

4. Consider the English sentence, "If some bakery sells stale bread and some hotel sells flat soda, then the only thing everyone likes is tea." This sentence has at least two meanings. Write two (logically different) first-order logic sentences that correspond to meanings that could be assigned to this sentence. Use the following predicates: $P(x)$ is *True* iff x is a person; $B(x)$ is *True* iff x is a bakery; $S_B(x)$ is *True* iff x sells stale bread; $H(x)$ is *True* iff x is a hotel; $S_S(x)$ is *True* iff x sells flat soda; $L(x, y)$ is *True* iff x likes y; and $T(x)$ is *True* iff x is tea.

5. Let P be the set of positive integers. Let $L = \{\mathtt{A}, \mathtt{B}, \ldots, \mathtt{Z}\}$ (i.e., the set of upper case characters in the English alphabet). Let T be the set of strings of one or more upper case English characters. Define the following predicates over those sets.

- For $x \in L$, $V(x)$ is *True* iff x is a vowel. (The vowels are A, E, I, O, and U.)
- For $x \in L$ and $n \in P$, $S(x, n)$ is *True* iff x can be written in n strokes.

A.4, A.5

- For $x \in L$ and $s \in T$, $O(x, s)$ is *True* iff x occurs in the string s.
- For $x, y \in L$, $B(x, y)$ is *True* iff x occurs before y in the English alphabet.
- For $x, y \in L$, $E(x, y)$ is *True* iff $x = y$.

Using these predicates, write each of the following statements as a sentence in first-order logic:

a. A is the only upper case English character that is a vowel and that can be written in three strokes but does not occur in the string STUPID.

b. There is an upper case English character strictly between K and R that can be written in one stroke.

6. Choose a set A and predicate P and then express the set $\{1, 4, 9, 16, 25, 36, \ldots\}$ in the form:

$$\{x \in A : P(x)\}.$$

7. Find a set that has a subset but no proper subset.

8. Give an example, other than one of the ones in the book, of a reflexive, symmetric, intransitive relation on the set of people.

9. Not equal (defined on the integers) is (circle all that apply): reflexive, symmetric, transitive.

A.6-A.9

11. Using the definition of $\equiv_p$ (equivalence modulo p) that is given in Example A.4, let R_p be a binary relation on $\mathbb{N}$, defined as follows, for any $p \geq 1$:

$$R_p = \{(a, b): a \equiv_p b\}.$$

So, for example R_3 contains $(0, 0)$, $(6, 9)$, $(1, 4)$, etc., but does not contain $(0, 1)$, $(3, 4)$, etc.

a. Is R_p an equivalence relation for every $p \geq 1$? Prove your answer.

b. If R_p is an equivalence relation, how many equivalence classes does it induce for a given value of p? What are they? (Any concise description is fine.)

c. Is R_p a partial order? A total order? Prove your answer.

A.11

A 12-13

12. Let $S = \{w \in \{\mathrm{a}, \mathrm{b}\}^*\}$. Define the relation *Substr* on the set S to be $\{(s, t) : s$ is a substring of $t\}$.

   a. Choose a small subset of *Substr* and draw it as a graph (in the same way that we drew the graph of Example A.5).

   b. Is *Substr* a partial order?

13. Let P be the set of people. Define the function:

$$father\text{-}of : P \rightarrow P.$$
$$father\text{-}of(x) = \text{the person who is } x\text{'s father}$$

   a. Is *father-of* one-to-one?

   b. Is it onto?

A 14.

Part d for turnin

14. Are the following sets closed under the following operations? If not, give an example that proves that they are not and then specify what the closure is.

   a. The negative integers under subtraction.

   b. The negative integers under division.

   c. The positive integers under exponentiation.

   d. The finite sets under Cartesian product.

   e. The odd integers under remainder, mod 3.

   f. The rational numbers under addition.

A14d: She should have said "the set of finite sets under …" I.e., is the Cartesian product of two finite sets always a finite set?

A 15

15. Give examples to show that:

   a. The intersection of two countably infinite sets can be finite.

   b. The intersection of two countably infinite sets can be countably infinite.

   c. The intersection of two uncountable sets can be finite.

   d. The intersection of two uncountable sets can be countably infinite.

   e. The intersection of two uncountable sets can be uncountable.

16. Let $R = \{(1, 2), (2, 3), (3, 5), (5, 7), (7, 11), (11, 13), (4, 6), (6, 8), (8, 9), (9, 10), (10, 12)\}$. Draw a directed graph representing R^*, the reflexive, transitive closure of R.

A17

17. Let $\mathbb{N}$ be the set of nonnegative integers. For each of the following sentences in first-order logic, state whether the sentence is valid, is not valid but is satisfiable, or is unsatisfiable. Assume the standard interpretation for $<$ and $>$. Assume that f could be any function on the integers. Prove your answer.

   a. $\forall x \in \mathbb{N}\, (\exists y \in \mathbb{N}\, (y < x))$

   b. $\forall x \in \mathbb{N}\, (\exists y \in \mathbb{N}\, (y > x))$

   c. $\forall x \in \mathbb{N}\, (\exists y \in \mathbb{N}\, f(x) = y)$

A18, A19

18. Let $\mathbb{N}$ be the set of nonnegative integers. Let A be the set of nonnegative integers x such that $x \equiv_3 0$. Show that $|\mathbb{N}| = |A|$.

A19c for turnin

19. What is the cardinality of each of the following sets? Prove your answer.

   a. $\{n \in \mathbb{N} : n \equiv_3 0\}$

   b. $\{n \in \mathbb{N} : n \equiv_3 0\} \cap \{n \in \mathbb{N} : n \text{ is prime}\}$.

   c. $\{n \in \mathbb{N} : n \equiv_3 0\} \cup \{n \in \mathbb{N} : n \text{ is prime}\}$

A 20

20. Prove that the set of rational numbers is countably infinite.