Analysis of algorithms

- **Issues:**
  - correctness
  - time efficiency
  - space efficiency
  - optimality
- **Approaches:**
  - theoretical analysis
  - empirical analysis

Best-case, average-case, worst-case

For some algorithms efficiency depends on form of input:

- **Worst case:** $C_{worst}(n)$ – maximum over inputs of size n
- **Best case:** $C_{best}(n)$ – minimum over inputs of size n
- **Average case:** $C_{avg}(n)$ – "average" over inputs of size n
  - Number of times the basic operation will be executed on typical input
  - NOT the average of worst and best case
  - Expected number of basic operations considered as a random variable under some assumption about the probability distribution of all possible inputs

Example: Sequential search

ALGORITHM *SequentialSearch*$(A[0..n-1], K)$

```
//Searches for a given value in a given array by sequential search
//Input: An array A[0..n − 1] and a search key K
//Output: The index of the first element of A that matches K
//        or −1 if there are no matching elements
i ← 0
while i < n and A[i] ≠ K do
    i ← i + 1
if i < n return i
else return −1
```

- **Worst case**
- **Best case**
- **Average case**

Asymptotic order of growth

A way of comparing functions that ignores constant factors and small input sizes

- **$O(g(n))$: class of functions $f(n)$ that grow no faster than $g(n)$**
- **$\Theta(g(n))$: class of functions $f(n)$ that grow at same rate as $g(n)$**
- **$\Omega(g(n))$: class of functions $f(n)$ that grow at least as fast as $g(n)$**

Big-oh

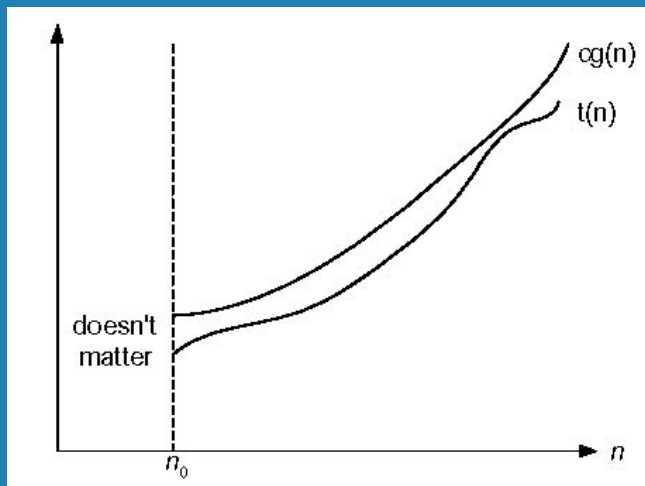

Figure 2.1 Big-oh notation: $t(n) \in O(g(n))$

Big-omega

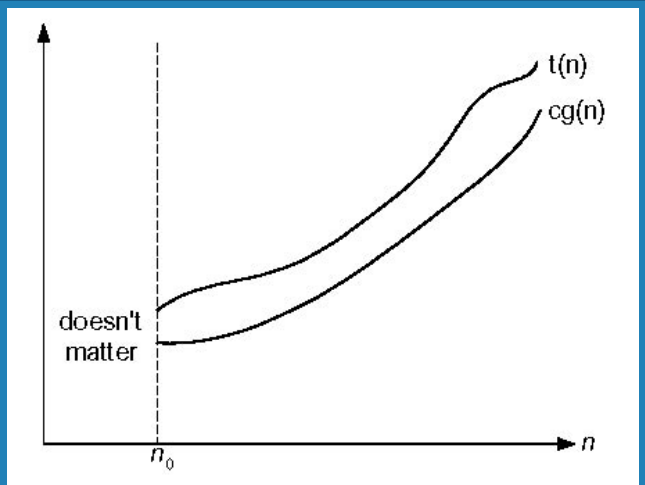

Fig. 2.2 Big-omega notation: $t(n) \in \Omega(g(n))$

Big-theta

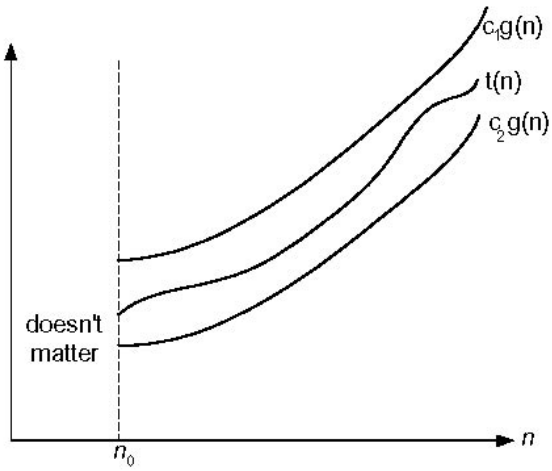

Figure 2.3 Big-theta notation: $t(n) \in \Theta(g(n))$

Establishing order of growth using the definition

Definition: $f(n)$ **is in** $O(g(n))$ **if order of growth of** $f(n) \leq$ **order of growth of** $g(n)$ **(within constant multiple), i.e., there exist positive constant** c **and non-negative integer** n_0 **such that**

$$f(n) \leq c\, g(n) \text{ for every } n \geq n_0$$

Examples:

- $10n$ **is** $O(n^2)$
- $5n+20$ **is** $O(n)$

Establishing order of growth using limits

$$\lim_{n\to\infty} T(n)/g(n) = \begin{cases} 0 & \text{order of growth of } T(n) < \text{order of growth of } g(n) \\ c > 0 & \text{order of growth of } T(n) = \text{order of growth of } g(n) \\ \infty & \text{order of growth of } T(n) > \text{order of growth of } g(n) \end{cases}$$

Examples:

- $10n$ vs. n^2
- $n(n+1)/2$ vs. n^2

Time efficiency of nonrecursive algorithms

General Plan for Analysis

- Decide on parameter n indicating *input size*
- Identify algorithm's *basic operation*
- Determine *worst*, *average*, and *best* cases for input of size n
- Set up a sum for the number of times the basic operation is executed
- Simplify the sum using standard formulas and rules (see Appendix A)

Useful summation formulas and rules

$\Sigma_{l \le i \le u} 1 = 1+1+\cdots+1 = u - l + 1$

In particular, $\Sigma_{1 \le i \le u} 1 = n - 1 + 1 = n \in \Theta(n)$

$\Sigma_{1 \le i \le n} i = 1+2+\cdots+n = n(n+1)/2 \approx n^2/2 \in \Theta(n^2)$

$\Sigma_{1 \le i \le n} i^2 = 1^2+2^2+\cdots+n^2 = n(n+1)(2n+1)/6 \approx n^3/3 \in \Theta(n^3)$

$\Sigma_{0 \le i \le n} a^i = 1 + a + \cdots + a^n = (a^{n+1} - 1)/(a - 1)$ for any $a \ne 1$

In particular, $\Sigma_{0 \le i \le n} 2^i = 2^0 + 2^1 + \cdots + 2^n = 2^{n+1} - 1 \in \Theta(2^n)$

$\Sigma(a_i \pm b_i) = \Sigma a_i \pm \Sigma b_i \quad \Sigma c a_i = c \Sigma a_i \quad \Sigma_{l \le i \le u} a_i = \Sigma_{l \le i \le m} a_i + \Sigma_{m+1 \le i \le u} a_i$

Example 1: Maximum element

ALGORITHM *MaxElement*$(A[0..n-1])$
//Determines the value of the largest element in a given array
//Input: An array $A[0..n-1]$ of real numbers
//Output: The value of the largest element in A
$maxval \leftarrow A[0]$
for $i \leftarrow 1$ **to** $n-1$ **do**
  if $A[i] > maxval$
    $maxval \leftarrow A[i]$
return $maxval$

Example 2: Element uniqueness problem

ALGORITHM *UniqueElements*$(A[0..n-1])$
//Determines whether all the elements in a given array are distinct
//Input: An array $A[0..n-1]$
//Output: Returns "true" if all the elements in A are distinct
// and "false" otherwise
for $i \leftarrow 0$ **to** $n-2$ **do**
  for $j \leftarrow i+1$ **to** $n-1$ **do**
    if $A[i] = A[j]$ **return false**
return true

Example 3: Matrix multiplication

ALGORITHM *MatrixMultiplication*$(A[0..n-1, 0..n-1], B[0..n-1, 0..n-1])$
//Multiplies two n-by-n matrices by the definition-based algorithm
//Input: Two n-by-n matrices A and B
//Output: Matrix $C = AB$
for $i \leftarrow 0$ **to** $n-1$ **do**
  for $j \leftarrow 0$ **to** $n-1$ **do**
    $C[i, j] \leftarrow 0.0$
    for $k \leftarrow 0$ **to** $n-1$ **do**
      $C[i, j] \leftarrow C[i, j] + A[i, k] * B[k, j]$
return C