MA/CSSE 473
Day 15

BFS

Topological Sort

Combinatorial Object Generation

MA/CSSE 473 Day 15

- **HW 6** due tomorrow, **HW 7** Friday, Exam next Tuesday
- **HW 8** has been updated for this term. Due Oct 8
- **ConvexHull implementation problem** due Day 27 (Oct 21); assignment is available now
  - Individual assignment (I changed my mind, but I am giving you 3.5 weeks to do it)
- Schedule page now has my projected dates for all remaining reading assignments and written assignments
  - Topics for future class days and details of assignments 9-17 still need to be updated for this term
- **Student Questions**
- DFS and BFS
- Topological Sort
- Combinatorial Object Generation - Intro

Q1

Recap: Pseudocode for DFS

```
ALGORITHM  DFS(G)
    //Implements a depth-first search traversal of a given graph
    //Input: Graph G = ⟨V, E⟩
    //Output: Graph G with its vertices marked with consecutive integers
    //in the order they've been first encountered by the DFS traversal
    mark each vertex in V with 0 as a mark of being "unvisited"
    count ← 0
    for each vertex v in V do
        if v is marked with 0
          dfs(v)

    dfs(v)
    //visits recursively all the unvisited vertices connected to vertex v by a path
    //and numbers them in the order they are encountered
    //via global variable count
    count ← count + 1; mark v with count
    for each vertex w in V adjacent to v do
        if w is marked with 0
          dfs(w)
```

Graph may not be connected, so we loop.

Backtracking happens when this loop ends (no more unmarked neighbors)

Analysis?

Q2

Notes on DFS

- DFS can be implemented with graphs represented as:
  - adjacency matrix: $\Theta(|V|^2)$
  - adjacency list: $\Theta(|V|+|E|)$
- Yields two distinct ordering of vertices:
  - order in which vertices are first encountered (pushed onto stack)
  - order in which vertices become dead-ends (popped off stack)
- Applications:
  - check connectivity, finding connected components
  - Is this graph acyclic?
  - finding articulation points, if any (advanced)
  - searching the state-space of problem for (optimal) solution (AI)

Q3

Breadth-first search (BFS)

- Visits graph vertices by moving across to all the neighbors of last visited vertex. Vertices closer to the start are visited early
- Instead of a stack, BFS uses a queue
- Level-order tree traversal is a special case of BFS
- “Redraws” graph in tree-like fashion (with tree edges and cross edges for undirected graph)

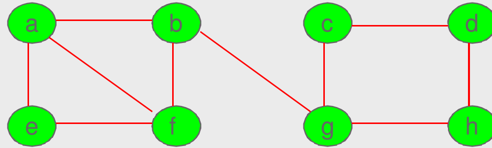
Pseudocode for BFS

ALGORITHM *BFS(G)*

```
//Implements a breadth-first search traversal of a given graph
//Input: Graph G = ⟨V, E⟩
//Output: Graph G with its vertices marked with consecutive integers
//in the order they have been visited by the BFS traversal
mark each vertex in V with 0 as a mark of being "unvisited"
count ← 0
for each vertex v in V do
    if v is marked with 0
        bfs(v)

bfs(v)
//visits all the unvisited vertices connected to vertex v by a path
//and assigns them the numbers in the order they are visited
//via global variable count
count ← count + 1;   mark v with count and initialize a queue with v
while the queue is not empty do
    for each vertex w in V adjacent to the front vertex do
        if w is marked with 0
            count ← count + 1;   mark w with count
            add w to the queue
    remove the front vertex from the queue
```

Example of BFS traversal of undirected graph



BFS traversal queue:

BFS tree:

Notes on BFS

- BFS has same efficiency as DFS and can be implemented with graphs represented as:
  - adjacency matrices: $\Theta(V^2)$
  - adjacency lists: $\Theta(|V|+|E|)$
- Yields single ordering of vertices (order added/deleted from queue is the same)
- Applications: same as DFS, but can also find shortest paths (smallest number of edges) from a vertex to all other vertices

DFS and BFS

TABLE 5.1 Main facts about depth-first search (DFS) and breadth-first search (BFS)

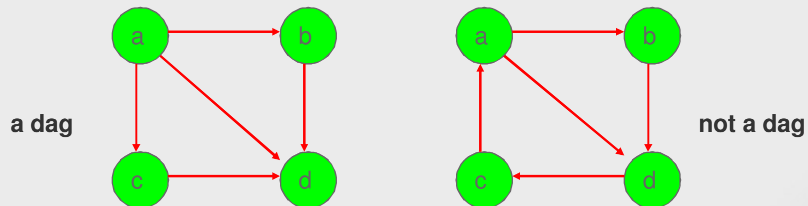
	DFS	BFS
Data structure	stack	queue
No. of vertex orderings	2 orderings	1 ordering
Edge types (undirected graphs)	tree and back edges	tree and cross edges
Applications	connectivity, acyclicity, articulation points	connectivity, acyclicity, minimum-edge paths
Efficiency for adjacent matrix	$\Theta(\|V^2\|)$	$\Theta(\|V^2\|)$
Efficiency for adjacent lists	$\Theta(\|V\| + \|E\|)$	$\Theta(\|V\| + \|E\|)$

Directed graphs

- In an undirected graph, each edge is a "two-way street".
  - The adjacency matrix is symmetric
- In an directed graph, each edge goes only one way.
  - (a,b) and (b,a) are separate edges.
  - One such edge can be in the graph without the other being there.

Dags and Topological Sorting

A *dag*: a directed acyclic graph, i.e. a directed graph with no (directed) cycles



Dags arise in modeling many problems that involve prerequisite constraints (construction projects, document version control, compilers)

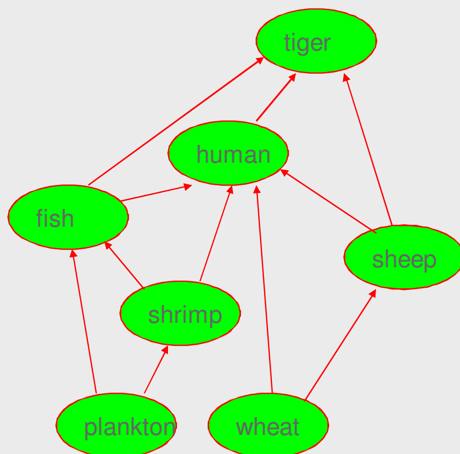
The vertices of a dag can be linearly ordered so that for every edge its starting vertex is listed before its ending vertex (**topological sort**).

A graph must be a dag in order for a topological sort of its vertices to be possible.

Q6-7

Topological Sorting Example

Order the following items in a food chain

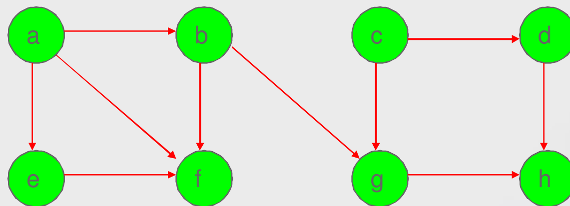


DFS-based Algorithm

DFS-based algorithm for topological sorting

- Perform DFS traversal, noting the order vertices are popped off the traversal stack
- Reversing order solves topological sorting problem
- Back edges encountered?→ NOT a dag!

Example:



Efficiency:

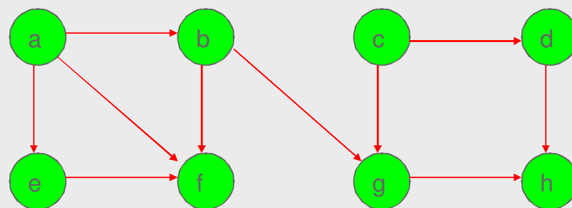
Q8

Source Removal Algorithm

Source removal algorithm

Repeatedly identify and remove a *source* (a vertex with no incoming edges) and all the edges incident to it until either no vertex is left (problem is solved) or there is no source among remaining vertices (not a dag)

Example:



Efficiency: same as efficiency of the DFS-based algorithm

Q9

Application: Spreadsheet program

- What is an allowable order of computation of the cells' values?

	A	B	C
1	=C4-7	4	=C4+6
2	=A3+A1-C4	=1+B1	=B1-A4
3	7	=A3*C2-B2	=B3+A3
4	=A1*B1*A2	=C2-A4	9

Q10-12

Cycles cause a problem!

	A	B	C
1	=C4-7	4	=C4+6
2	=A3+A1-C4	=1+B1	=B1-A4
3	7	=A3*C2-B2	=B3+A3
4	=A1*B1*A2	=C2-A4	9

(We may not get to this today)

Permutations

Subsets

COMBINATORIAL OBJECT GENERATION

Combinatorial Object Generation

- Generation of permutations, combinations, subsets.
- This is a big topic in CS
- We will just scratch the surface of this subject.
  - Permutations of a list of elements (no duplicates)
  - Subsets of a set

Permutations

- We generate all permutations of the numbers 1..n.
  - Permutations of any other collection of n distinct objects can be obtained from these by a simple mapping.
- How would a "decrease by 1" approach work?
  - Find all permutations of 1.. n-1
  - Insert n into each position of each such permutation
  - We'd like to do it in a way that minimizes the change from one permutation to the next.
  - It turns out we can do it so that we always get the next permutation by swapping two adjacent elements.

First approach we might think of

- for each permutation of 1..n-1
  - for i=0..n-1
    - insert n in position i
- That is, we do the insertion of n into each smaller permutation from left to right each time
- However, to get "minimal change", we alternate:
  - Insert n L-to-R in one permutation of 1..n-1
  - Insert n R-to-L in the next permutation of 1..n-1
  - Etc.

Example

- Bottom-up generation of permutations of 123

start	1		
insert 2 into 1 right to left	12	21	
insert 3 into 21 right to left	123	132	312
insert 3 into 21 left to right	321	231	213

FIGURE 5.12 Generating permutations bottom up

- Example: Do the first few permutations for n=4

Johnson-Trotter Approach

- integrates the insertion of n with the generation of permutations of 1..n-1
- Does it by keeping track of which direction each number is currently moving

$$\overrightarrow{3}\overleftarrow{2}\overrightarrow{4}\overleftarrow{1}$$

  The number k is **mobile** if its arrow points to an adjacent element that is smaller than itself.
- In this example, 4 and 3 are mobile
- We exchange the largest mobile number with its neighbor

Johnson-Trotter Driver

```
def main():
    p = Permutation(4)
    list = []
    next = p.next()
    while next:
        list += [next]
        next =  p.next()
    print list
```

Johnson-Trotter background code

```
left = - 1   # equivalent to the left- and
right = 1    # right-pointing arrows in the book

def swap(list1, list2, i, j):
    "Swap positions i and j in both lists"
    list1[i], list1[j] = list1[j], list1[i]
    list2[i], list2[j] = list2[j], list2[i]


class Permutation:
    "Set current to the unpermuted list, and all directions pointing left"
    def __init__(self, n):
        self.current = range(1, n + 1)
        self.direction = [left] * n
        self.n = n
        self.more = True # This is not the last permutation.
```

Johnson-Trotter major methods

```
def isMobile(self, k):
    ''' An element of a permutation is mobile if its direction "arrow"
        points to an element with a smaller value.'''
    return k + self.direction[k] in range(self.n) and \
           self.current[k + self.direction[k]] < self.current[k]

def next(self):
    "return current permutation and calculate next one"
    if not self.more:
        return False
    returnValue = [self.current[i] for i in range(self.n)]

    largestMobile = 0
    for i in range(self.n):
        if self.isMobile(i) and self.current[i] > largestMobile:
                largestMobile = self.current[i]
                largePos = i

    if largestMobile == 0:
        self.more = False  # This is the last permutation
    else:
        swap(self.current, self.direction,
             largePos, largePos + self.direction[largePos])
        for i in range(self.n):
            if self.current[i] > largestMobile:
                self.direction[i] *= - 1

    return "".join([str(v) for v in returnValue])
```

Lexicographic Permutation Generation

- Generate the permutations in "natural" order.
- Let's do it recursively.