MA/CSSE 473 Day 28

Optimal BSTs

Dynamic Programming Example

OPTIMAL BINARY SEARCH TREES

Warmup: Optimal linked list order

- Suppose we have n distinct data items $x_1, x_2, \ldots, x_n$ in a linked list.
- Also suppose that we know the probabilities $p_1, p_2, \ldots, p_n$ that each of these items is the item we'll be searching for.
- Questions we'll attempt to answer:
  - What is the expected number of probes before a successful search completes?
  - How can we minimize this number?
  - What about an unsuccessful search?

Examples

- $p_i = 1/n$ for each i.
  - What is the expected number of probes?
- $p_1 = ½, p_2 = ¼, \ldots, p_{n-1} = 1/2^{n-1}, p_n = 1/2^{n-1}$
  - expected number of probes:

$$\sum_{i=1}^{n-1} \frac{i}{2^i} + \frac{n}{2^{n-1}} = 2 - \frac{1}{2^{n-1}} < 2$$

- What if the same items are placed into the list in the opposite order?

$$\sum_{i=2}^{n} \frac{i}{2^{n+1-i}} + \frac{1}{2^{n-1}} = n - 1 + \frac{1}{2^{n-1}}$$

- The next slide shows the evaluation of the last two summations in Maple.
  - Good practice for you? prove them by induction

Calculations for previous slide

```
> sum(i/2^i, i=1..n-1)+ n/2^(n-1);
```

$$-2\left(\frac{1}{2}\right)^{n} n-2\left(\frac{1}{2}\right)^{n}+2+\frac{n}{2^{(n-1)}}$$

```
> simplify(%);
```

$$-2^{(1-n)}+2$$

```
> sum(i/2^(n+1-i), i=2..n) + 1/2^(n-1);
```

$$n-1+\frac{1}{2^{(n-1)}}$$

What if we don't know the probabilities?

1. Sort the list so we can at least improve the average time for unsuccessful search
2. Self-organizing list:
   - Elements accessed more frequently move toward the front of the list; elements accessed less frequently toward the rear.
   - Strategies:
     - Move ahead one position (exchange with previous element)
     - Exchange with first element
     - Move to Front (only efficient if the list is a linked list)

- What we are actually likely to know is frequencies in previous searches.
- Our best estimate of the probabilities will be proportional to the frequencies, so we can use frequencies instead of probabilities.

Optimal Binary Search Trees

- Suppose we have n distinct data keys K_1, K_2, …, K_n (in increasing order) that we wish to arrange into a Binary Search Tree
- Suppose we know the probabilities that a successful search will end up at K_i and the probabilities that the key in an unsuccessful search will be larger than K_i and smaller than K_{i+1}
- This time the expected number of probes for a successful or unsuccessful search depends on the shape of the tree and where the search ends up
  - Formula?
- Guiding principle for optimization?

Example

- For now we consider only successful searches, with probabilities A(0.2), B(0.3), C(0.1), D(0.4).
- What would be the worst-case arrangement for the expected number of probes?
  - For simplicity, we'll multiply all of the probabilities by 10 so we can deal with integers.
- Try some other arrangements: Opposite, Greedy, Better, Best?
- Brute force: Try all of the possibilities and see which is best. How many possibilities?

Aside: How many possible BST's

- Given distinct keys $K_1 < K_2 < \ldots < K_n$, how many different Binary Search Trees can be constructed from these values?
- Figure it out for n=2, 3, 4, 5
- Write the recurrence relation

Aside: How many possible BST's

- Given distinct keys $K_1 < K_2 < \ldots < K_n$, how many different Binary Search Trees can be constructed from these values?
- Figure it out for n=2, 3, 4, 5
- Write the recurrence relation
- Solution is the **Catalan number** c(n)

When n=20, c(n) is almost 10^{10}

$$c(n) = \binom{2n}{n}\frac{1}{n+1} = \frac{(2n)!}{n!(n+1)!} = \prod_{k=2}^{n}\frac{n+k}{k} \approx \frac{4^n}{n^{3/2}\sqrt{\pi}}$$

- Verify for n = 2, 3, 4, 5.

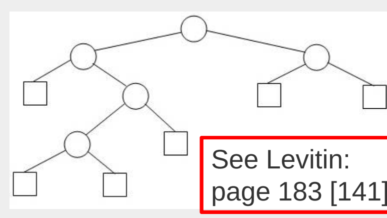
Wikipedia Catalan article has five different proofs of $c(n) = \binom{2n}{n}\frac{1}{n+1}$

Optimal Binary Search Trees

- Suppose we have n distinct data keys K_1, K_2, …, K_n (in increasing order) that we wish to arrange into a Binary Search Tree
- This time the expected number of probes for a successful or unsuccessful search depends on the shape of the tree and where the search ends up
- This discussion follows Reingold and Hansen, *Data Structures*. **An excerpt on optimal static BSTS is posted on Moodle.** I use a_i and b_i where Reingold and Hansen use α_i and β_i

Recap: Extended binary search tree

- It's simplest to describe this problem in terms of an **extended binary search tree** (EBST): a BST enhanced by drawing "external nodes" in place of all of the null pointers in the original tree



See Levitin: page 183 [141]

- Formally, an Extended Binary Tree (EBT) is either
  - an external node, or
  - an (internal) root node and two EBTs T_L and T_R
- **In diagram, Circles = internal nodes, Squares = external nodes**
- It's an alternative way of viewing a binary tree
- **The external nodes stand for places where an unsuccessful search can end or where an element can be inserted**
- An EBT with n internal nodes has ___ external nodes (We proved this by induction earlier in the term)

What contributes to the expected number of probes?

- Frequencies, depth of node
- For successful search, number of probes is __one more than__ the depth of the corresponding internal node
- For unsuccessful, number of probes is __equal to__ the depth of the corresponding external node

Optimal BST Notation

- Keys are K_1, K_2, …, K_n
- Let v be the value we are searching for
- For i= 1, …,n, let a_i be the probability that v is key K_i
- For i= 1, …,n-1, let b_i be the probability that $K_i < v < K_{i+1}$
  - Similarly, let b_0 be the probability that $v < K_1$, and b_n the probability that $v > K_n$
- Note that $$\sum_{i=1}^{n} a_i + \sum_{i=0}^{n} b_i = 1$$
- We can also just use *frequencies* instead of *probabilities* when finding the optimal tree (and divide by their sum to get the probabilities if we ever need them). That is what we will do in anexample.
- Should we try exhaustive search of all possible BSTs?

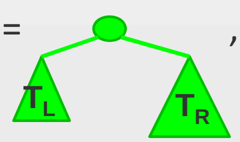
What not to measure

- What about external path length and internal path length?
- These are too simple, because they do not take into account the frequencies.
- We need *weighted* path lengths.

Weighted Path Length

$$C(T) = \sum_{i=1}^{n} a_i[1 + depth(x_i)] + \sum_{i=0}^{n} b_i[depth(y_i)]$$

Note: $y_0, \ldots, y_n$ are the external nodes of the tree

- If we divide this by $\Sigma a_i + \Sigma b_i$ we get the expected number of probes.
- We can also define it recursively:
- C(☐) = 0. If T =  , then

  $C(T) = C(T_L) + C(T_R) + \Sigma a_i + \Sigma b_i$, where the summations are over all a_i and b_i for nodes in T
- It can be shown by induction that these two definitions are equivalent (a homeworkproblem,).

Example

- Frequencies of vowel occurrence in English
- : A, E, I, O, U
- a's: 32,   42,   26,   32,   12
- b's: 0,   34,   38,   58,   95,   21
- Draw a couple of trees (with E and I as roots), and see which is best. (sum of a's and b's is 390).

Strategy

- We want to minimize the weighted path length
- Once we have chosen the root, the left and right subtrees must themselves be optimal EBSTs
- We can build the tree from the bottom up, keeping track of previously-computed values

Intermediate Quantities

- Cost: Let C_{ij} (for $0 \le i \le j \le n$) be the cost of an optimal tree (not necessarily unique) over the frequencies $b_i, a_{i+1}, b_{i+1}, \ldots a_j, b_j$. Then
- $C_{ii} = 0$, and $C_{ij} = \min_{i<k\le j}(C_{i,k-1} + C_{kj}) + \sum_{t=i}^{j} b_t + \sum_{t=i+1}^{j} a_t$
- This is true since the subtrees of an optimal tree must be optimal
- To simplify the computation, we define
- $W_{ii} = b_i$, and $W_{ij} = W_{i,j-1} + a_j + b_j$ for $i<j$.
- Note that $W_{ij} = b_i + a_{i+1} + \ldots + a_j + b_j$, and so
- $C_{ii} = 0$, and $C_{ij} = W_{ij} + \min_{i<k\le j}(C_{i,k-1} + C_{kj})$
- Let R_{ij} (root of best tree from i to j) be a value of k that minimizes $C_{i,k-1} + C_{kj}$ in the above formula

Code

```
# initialize the main diagonal
for i in range(n + 1):
    R[i][i] = i
    W[i][i] = b[i]
    # Draw this cell of the table in the given window.
    drawSquare(i, i, W[i][i], C[i][i], R[i][i], win, indent, squareSize)
# Now populate each of the n upper diagonals:
for d in range(1, n+1): # fill in this diagonal
    # The previous diagonals are already filled in.
    for i in range(n - d + 1):
        j = i + d; # on the dth diagonal, j - i = d
        opt = i + 1 # until we find a better one
        for k in range(i+2, j+1):
            if C[i][k-1]+C[k][j] < C[i][opt-1]+C[opt][j]:
                opt = k
        R[i][j] = opt
        W[i][j] = W[i][j-1] + a[j] + b[j]
        C[i][j] = C[i][opt-1] + C[opt][j] + W[i][j]
        # Draw this cell of the table in the given window.
        drawSquare(i, j, W[i][j], C[i][j], R[i][j], win, indent, squareSize)
```

Results

R00: 0 W00: 0 C00: 0	R01: 1 W01: 66 C01: 66	R02: 2 W02: 146 C02: 212	R03: 2 W03: 230 C03: 418	R04: 3 W04: 357 C04: 754	R05: 4 W05: 390 C05: 936
	R11: 1 W11: 34 C11: 0	R12: 2 W12: 114 C12: 114	R13: 3 W13: 198 C13: 312	R14: 3 W14: 325 C14: 624	R15: 4 W15: 358 C15: 798
		R22: 2 W22: 38 C22: 0	R23: 3 W23: 122 C23: 122	R24: 4 W24: 249 C24: 371	R25: 4 W25: 282 C25: 532
			R33: 3 W33: 58 C33: 0	R34: 4 W34: 185 C34: 185	R35: 4 W35: 218 C35: 346
				R44: 4 W44: 95 C44: 0	R45: 5 W45: 128 C45: 128
					R55: 5 W55: 21 C55: 0

How to construct the optimal tree?

Analysis of the algorithm?

- Constructed by diagonals, from main diagonal upward
- What is the optimal tree?

Running time

- Most frequent statement is the comparison
  if C[i][k-1]+C[k][j] < C[i][opt-1]+C[opt][j]:
- How many times does it execute:

$$\sum_{d=1}^{n}\sum_{i=0}^{n-d}\sum_{k=i+2}^{i+d}1$$

```
simplify(sum(sum(sum(1,k=i+2..i+d),i=0..n-d),d=1..n));
```

$$-\frac{1}{6}n+\frac{1}{6}n^3$$

Do what seems best at the moment …

GREEDY ALGORITHMS

Greedy algorithms

- Whenever a choice is to be made, pick the one that seems optimal for the moment, without taking future choices into consideration
  - Once each choice is made, it is irrevocable
- For example, a greedy Scrabble player will simply maximize her score for each turn, never saving any “good” letters for possible better plays later
  - Doesn’t necessarily optimize score for entire game
- Greedy works well for the "optimal linked list with known search probabilities" problem, and reasonably well for the "optimal BST" problem
  - But does not necessarily produce an optimal tree

Q7

Greedy Chess

- Take a piece or pawn whenever you will not lose a piece or pawn (or will lose one of lesser value) on the next turn
- Not a good strategy for this game either

Greedy Map Coloring

- On a planar (i.e., 2D Euclidean) connected map, choose a region and pick a color for that region
- Repeat until all regions are colored:
  - Choose an uncolored region R that is adjacent[1] to at least one colored region
    - If there are no such regions, let R be any uncolored region
  - Choose a color that is different than the colors of the regions that are adjacent to R
  - Use a color that has already been used if possible
- The result is a valid map coloring, not necessarily with the minimum possible number of colors

[1] Two regions are *adjacent* if they have a common edge

Spanning Trees for a Graph

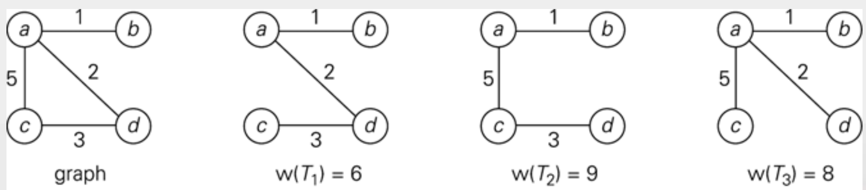
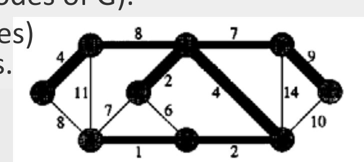


FIGURE 9.1 Graph and its spanning trees; T_1 is the minimum spanning tree

Minimal Spanning Tree (MST)

- Suppose that we have a connected network G (a graph whose edges are labeled by numbers, which we call **weights**)
- We want to find a tree T that
  - spans the graph (i.e. contains all nodes of G).
  - minimizes (among all spanning trees) the sum of the weights of its edges.
- Is this MST unique?



- One approach: Generate all spanning trees and determine which is minimum
- Problems:
  - The number of trees grows exponentially with N
  - Not easy to generate
  - Finding a MST directly is simpler and faster

More details soon

Huffman's algorithm

- Goal: We have a message that co9ntains n different alphabet symbols. Devise an encoding for the symbols that minimizes the total length of the message.
- Principles: More frequent characters have shorter codes. No code can be a prefix of another.
- Algorithm: Build a tree form which the codes are derived. Repeatedly join the two lowest-frequency trees into a new tree.

Q8-10