MA/CSSE 473
Day 29

Optimal BST
Conclusion

Then student questions about the exam

Dynamic Programming Example

OPTIMAL BINARY SEARCH TREES

Recap: Optimal Binary Search Trees

- Suppose we have n distinct data keys K_1, K_2, …, K_n (in increasing order) that we wish to arrange into a Binary Search Tree
- This time the expected number of probes for a successful or unsuccessful search depends on the shape of the tree and where the search ends up
- Guiding principle for optimization?
- This discussion follows Reingold and Hansen, *Data Structures*. **An excerpt on optimal static BSTS is posted on Moodle.** I use a_i and b_i where Reingold and Hansen use α_i and β_i

What contributes to the expected number of probes?

- Frequencies, depth of node
- For successful search, number of probes is __one more than__ the depth of the corresponding internal node
- For unsuccessful, number of probes is __equal to__ the depth of the corresponding external node

Optimal BST Notation

- Keys are $K_1, K_2, \ldots, K_n$, in internal nodes $x_1, x_2, \ldots, x_n$
- Let v be the value we are searching for
- For i= 1, …,n, let a_i be the probability that v is key K_i
- For i= 1, …,n-1, let b_i be the probability that $K_i < v < K_{i+1}$
  - Similarly, let b_0 be the probability that $v < K_1$, and b_n the probability that $v > K_n$
  - Each b_i is associated with external node y_i
- Note that $$\sum_{i=1}^{n} a_i + \sum_{i=0}^{n} b_i = 1$$
- We can also just use *frequencies* instead of *probabilities* when finding the optimal tree (and divide by their sum to get the probabilities if we ever need them). That is what we will do in an example.
- Should we try exhaustive search of all possible BSTs?

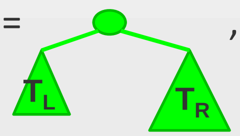
What not to measure

- What about external path length and internal path length?
- These are too simple, because they do not take into account the frequencies.
- We need *weighted* path lengths.

Weighted Path Length

$$C(T) = \sum_{i=1}^{n} a_i[1 + depth(x_i)] + \sum_{i=0}^{n} b_i[depth(y_i)]$$

Note: $y_0, \ldots, y_n$ are the external nodes of the tree

- If we divide this by $\Sigma a_i + \Sigma b_i$, we get the expected number of probes.
- We can also define C recursively:
- C(□) = 0. If T = 

, then

  $C(T) = C(T_L) + C(T_R) + \Sigma a_i + \Sigma b_i$, where the summations are over all a_i and b_i for nodes in T
- It can be shown by induction that these two definitions are equivalent (a homework problem).

Example

- Frequencies of vowel occurrence in English
- : A, E, I, O, U
- a's: 32, 42, 26, 32, 12
- b's: 0, 34, 38, 58, 95, 21
- Draw a tree (with E as root), and see which is best. (sum of a's and b's is 390).

Strategy

- We want to minimize the weighted path length
- Once we have chosen the root, the left and right subtrees must themselves be optimal EBSTs
- We can build the tree from the bottom up, keeping track of previously-computed values

Intermediate Quantities

- Cost: Let C_{ij} (for $0 \le i \le j \le n$) be the cost of an optimal tree (not necessarily unique) over the frequencies b_i, a_{i+1}, b_{i+1}, …a_j, b_j. Then
- $C_{ii} = 0$, and $C_{ij} = \min_{i<k\le j}(C_{i,k-1} + C_{kj}) + \sum_{t=i}^{j} b_t + \sum_{t=i+1}^{j} a_t$
- This is true since the subtrees of an optimal tree must be optimal
- To simplify the computation, we define
- $W_{ii} = b_i$, and $W_{ij} = W_{i,j-1} + a_j + b_j$ for $i<j$.
- Note that $W_{ij} = b_i + a_{i+1} + \ldots + a_j + b_j$, and so
- $C_{ii} = 0$, and $C_{ij} = W_{ij} + \min_{i<k\le j}(C_{i,k-1} + C_{kj})$
- Let R_{ij} (root of best tree from i to j) be a value of k that minimizes $C_{i,k-1} + C_{kj}$ in the above formula

Code

```
# initialize the main diagonal
for i in range(n + 1):
    R[i][i] = i
    W[i][i] = b[i]
    # Draw this cell of the table in the given window.
    drawSquare(i, i, W[i][i], C[i][i], R[i][i], win, indent, squareSize)
# Now populate each of the n upper diagonals:
for d in range(1, n+1): # fill in this diagonal
    # The previous diagonals are already filled in.
    for i in range(n - d + 1):
        j = i + d; # on the dth diagonal, j - i = d
        opt = i + 1 # until we find a better one
        for k in range(i+2, j+1):
            if C[i][k-1]+C[k][j] < C[i][opt-1]+C[opt][j]:
                opt = k
        R[i][j] = opt
        W[i][j] = W[i][j-1] + a[j] + b[j]
        C[i][j] = C[i][opt-1] + C[opt][j] + W[i][j]
        # Draw this cell of the table in the given window.
        drawSquare(i, j, W[i][j], C[i][j], R[i][j], win, indent, squareSize)
```

Results

R00: 0 W00: 0 C00: 0	R01: 1 W01: 66 C01: 66	R02: 2 W02: 146 C02: 212	R03: 2 W03: 230 C03: 418	R04: 3 W04: 357 C04: 754	R05: 4 W05: 390 C05: 936
	R11: 1 W11: 34 C11: 0	R12: 2 W12: 114 C12: 114	R13: 3 W13: 198 C13: 312	R14: 3 W14: 325 C14: 624	R15: 4 W15: 358 C15: 798
		R22: 2 W22: 38 C22: 0	R23: 3 W23: 122 C23: 122	R24: 4 W24: 249 C24: 371	R25: 4 W25: 282 C25: 532
			R33: 3 W33: 58 C33: 0	R34: 4 W34: 185 C34: 185	R35: 4 W35: 218 C35: 346
				R44: 4 W44: 95 C44: 0	R45: 5 W45: 128 C45: 128
					R55: 5 W55: 21 C55: 0

- Constructed by diagonals, from main diagonal upward
- What is the optimal tree?

How to construct the optimal tree?

Analysis of the algorithm?

Running time

- Most frequent statement is the comparison
  if C[i][k-1]+C[k][j] < C[i][opt-1]+C[opt][j]:
- How many times does it execute: $\sum_{d=1}^{n}\sum_{i=0}^{n-d}\sum_{k=i+2}^{i+d} 1$

```
simplify(sum(sum(sum(1,k=i+2..i+d),i=0..n-d),d=1..n));
```

$$-\frac{1}{6}n+\frac{1}{6}n^3$$