MA/CSSE 473 Day 16

Answers to your questions

Divide and Conquer

Closest Points

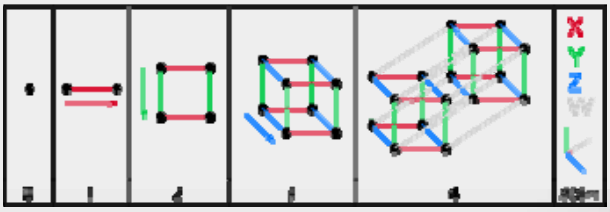
Convex Hull intro

Exercise from last time

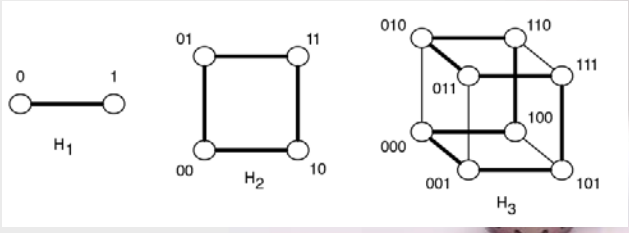
- Which permutation follows each of these in lexicographic order?
  - 183647520 471638520
  - Try to write an algorithm for generating the next permutation, with only the current permutation as input.
- If the lexicographic permutations of the numbers [0, 1, 2, 3, 4, 5] are numbered starting with 0, what is the number of the permutation 14032?
  - General form? How to calculate efficiency?
- In the lexicographic ordering of permutations of [0, 1, 2, 3, 4, 5], which permutation is number 541?
  - How to calculate efficiently?

Gray Code and Hamiltonian Cycles

- A Hamiltonian cycle in an undirected graph is …
- Hypercubes (picture is from Wikipedia):



- Binary-reflected Gray Code is a Hamiltonian Cycle of a Hypercube:



DIVIDE AND CONQUER

Divide-and-conquer algorithms

- Definition
- List examples seen in prior courses or so far in this course

Today: Closest Points, Convex Hull

DIVIDE AND CONQUER ALGORITHMS

Divide-and-conquer algorithms

- Definition
- Examples seen prior to this course or so far in this course

Q1

Closest Points problem

- Given a set, S, of N points in the xy-plane, find the minimum distance between two points in S.
- Running time for brute force algorithm?
- Next we examine a divide-and-conquer approach.

Closest Points "divide" phase

- S is a set of N points in the xy-plane
- For simplicity, we assume $N = 2^k$ for some k.
  (Otherwise use floor and ceiling functions)
- Sort the points by x-coordinate
  - If two points have the same x-coordinate, order them by y-coordinate
  - If we use merge sort, the worst case is $\Theta(N \log N)$
- Let c be the median x-value of the points
- Let S_1 be $\{(x, y): x \leq c\}$, and S_2 be $\{(x, y): x \geq c\}$
  - adjust so we get exactly N/2 points in each subset

Closest Points "conquer" phase

- Assume that the points of S are sorted by x-coordinate, then by y-coordinate if x's are equal
- Let d_1 be the minimum distance between two points in S_1 (the set of "left half" points)
- Let d_2 be the minimum distance between two points in S_2 (the set of "right half" points)
- Let $d = \min(d_1, d_2)$. Is d the minimum distance for S?
- What else do we have to consider?
- Suppose we needed to compare every point in S_1 to every point in S_2. What would the running time be?
- How can we avoid doing so many comparisons?

Reference: The Master Theorem

- The Master Theorem for Divide and Conquer recurrence relations:
- Consider the recurrence $T(n) = aT(n/b) + f(n)$, $T(1)=c$, where $f(n) = \Theta(n^k)$ and $k \geq 0$,
- The solution is
  - $\Theta(n^k)$ if $a < b^k$
  - $\Theta(n^k \log n)$ if $a = b^k$
  - $\Theta(n^{\log_b a})$ if $a > b^k$

For details, see Levitin pages 483-485 or Weiss section 7.5.3.

Grimaldi's Theorem 10.1 is a special case of the Master Theorem.

We will use this theorem often. You should review its proof soon (Weiss's proof is a bit easier than Levitin's).

After recursive calls on S_1 and S_2

$d = \min(d_1, d_2)$.

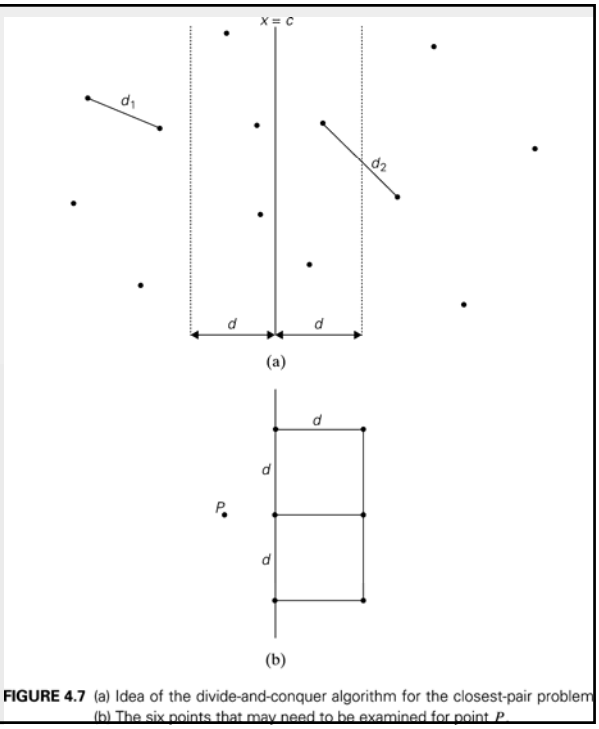


FIGURE 4.7 (a) Idea of the divide-and-conquer algorithm for the closest-pair problem (b) The six points that may need to be examined for point P.

Convex Hull Problem

- Again, sort by x-coordinate, with tie going to larger y-coordinate.

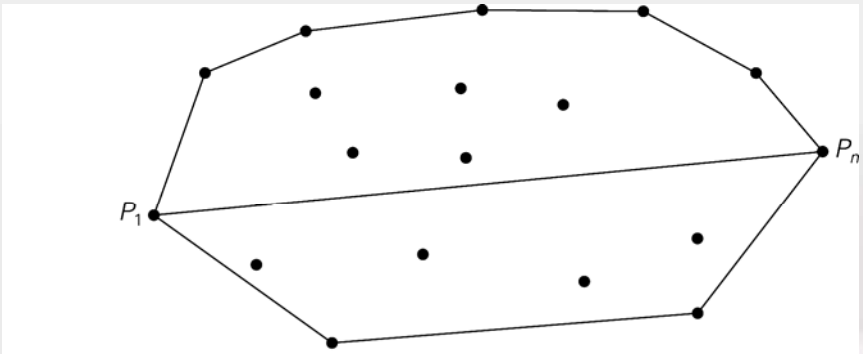


FIGURE 4.8 Upper and lower hulls of a set of points

Recursive calculation of Upper Hull

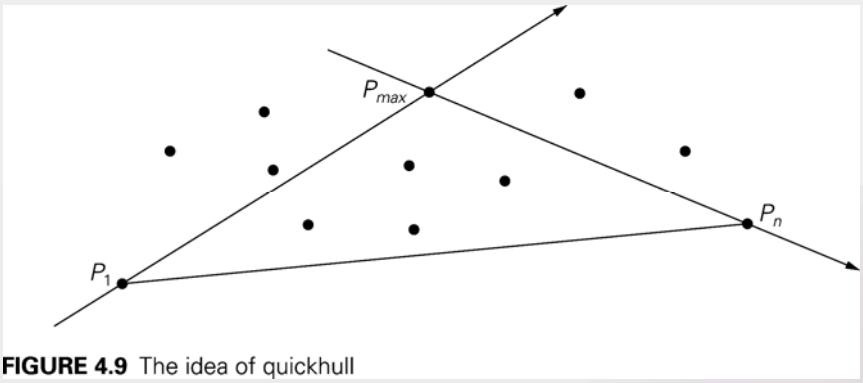


FIGURE 4.9 The idea of quickhull

Simplifying the Calculations

We can simplify two things at once:

- Finding the distance of P from line P_1P_2, and
- Determining whether P is "to the left" of P_1P_2
  - The area of the triangle through $P_1=(x_1,y_1)$, $P_2=(x_2,y_2)$, and $P_3=(x_3,y_e)$ is ½ of the absolute value of the determinant

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = x_1y_2 + x_3y_1 + x_2y_3 - x_3y_2 - x_2y_1 - x_1y_3$$

    - For a proof of this property, see http://mathforum.org/library/drmath/view/55063.html
    - How do we use this to calculate distance from P to the line?
  - The sign of the determinant is positive if the order of the three points is clockwise, and negative if it is counter-clockwise
    - Clockwise means that P_3 is "to the left" of directed line segment P_1P_2
- Speeding up the calculation

Efficiency of quickhull algorithm

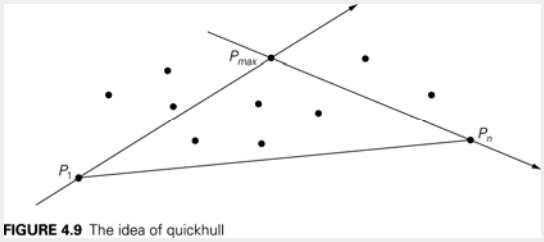


FIGURE 4.9 The idea of quickhull

- What arrangements of points give us worst case behavior?
- Average case is much better. Why?