Mathematical Induction

What it is?

Why is it a legitimate proof method?

How to use it?

Induction Proofs Outline

- Foundation:
  logic background
  the well-ordering principle.
- What is the principle of Mathematical induction?
- Induction in Action
- How **not** to do induction proofs
- Strong Induction

Some Logic Background

- **Implication**: A→B, where A and B are boolean values. **If A, then B**
  - If it rains today, I will use my umbrella.
    - When is this statement true?
      - When I use my umbrella
      - When it does not rain
    - When is it false?
      - Only when it rains and I do not use my umbrella.

A	B	A→B
T	T	T
T	F	F
F	T	T
F	F	T

Implication: A→B

A	B	A→B
T	T	T
T	F	F
F	T	T
F	F	T

- Conclusions:
  - If we know that **A→B** is true, and **B** is false, then **A** must be …
  - Another expression that is equivalent to A→B:
    - ¬ A OR B (think about the truth table for ¬A OR B)
  - What similar expression is equivalent to ¬(A→B)?

If **A→B** is true, and **B** is false, then **A** must be false
Answer to last question: A AND ¬ B

Contrapositive of A→B

A	B	A→B	¬B	¬A	¬B → ¬A
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

- ¬B → ¬A is called the **contrapositive** of A→B
- Notice that the third and sixth columns of the truth table are the same.
- Thus an implication is true if and only if its contrapositive is true.

Contradictions

- If A is a boolean value, the value of the expression **A AND ¬A** is ______. This expression is known as a **contradiction**.
- Putting this together with what we saw previously, if B → (A AND ¬A) **is True, what can we say about B?**
- **This is the basis for "proof by contradiction".**
  - **To show that B is true, we find an A for which we can show that ¬B → (A AND ¬A) is true.**
  - **This is the approach we will use in our proof that Mathematical induction works.**

Answer to question: False:

The Well-ordering principle

- It's an axiom, not something that we can prove.
- WOP: Every non-empty set of non-negative integers has a smallest element.
- Note the importance of "non-empty", "non-negative", and "integers".
  - The empty set does not have a smallest element.
  - A set with no lower bound (such as the set of all integers) does not have a smallest element.
    - In the statement of WOP, we can replace "positive" with "has a lower bound"
  - Unlike integers, a set of rational numbers can have a lower bound but no smallest member: {1/3, 1/5, 1/7, 1/9, … }
- Assuming the well-ordering principle, we will prove that the principle of mathematical induction is true.

What kind of things do we try to prove *via* Mathematical Induction?

- In this course, it will usually be a property of positive integers (or non-negative integers, or integers larger than some specific number).
- A ***property*** **p(n)** is a boolean statement about the integer *n*. [**p**: int $\rightarrow$ boolean]
  - Example: p(n) could be "n is an even number".
  - Then p(4) is true, but p(3) is false.
- If we believe that some property **p** is true for **all** positive integers, induction gives us a way of proving it.

The Principle of Mathematical Induction

- To prove that p(n) is true for all n $\geq n_0$:
  - Show that p(n_0) is true.
  - Show that **for all** k $\geq n_0$,

    p(k) implies p(k+1).

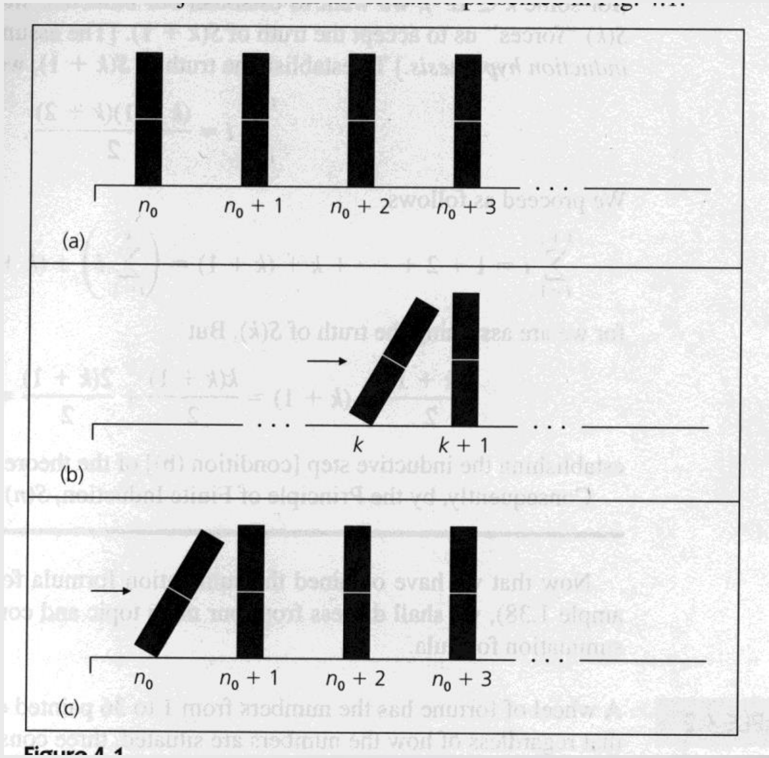
    I.e, show that **whenever** p(k) is true, then

    p(k+1) is true also.

Why does induction work? (Informal look)

- To prove that p(n) is true for all $n \geq n_0$:
  - Show that $p(n_0)$ is true.
  - Show that for all $k \geq n_0$, p(k) implies p(k+1).

    I.e, if p(k) is true, then p(k+1) is true also



First bullet, third picture.
Second bullet, second picture.
They all fall down!

Pictures from Ralph Grimaldi's discrete math book.

Why does induction work?

- Next we focus on a formal proof of this, because:
  - Some people may not be convinced by the informal one
  - The proof itself illustrates some important proof techniques

Proof that induction works (Overview)

- Let p be a property (p: int → boolean).
- Hypothesis:
  a) $p(n_0)$ is true.
  b) For all $k \geq n_0$, p(k) implies p(k+1). I.e, if p(k) is true, then p(k+1) is true also
- Desired Conclusion: If n is any integer with $n \geq n_0$, then p(n) is true. **If we can prove this, then induction is a legitimate proof method.**

- Proof that the conclusion follows from the hypothesis:
- Let S be the set $\{n \geq n_0 : p(n) \text{ is false}\}$.
- It suffices to show that S is empty.
- We do it by contradiction.
  - Assume that S is non-empty and show that this leads to a contradiction.

Proof that induction works (more details)

- Hypothesis:
  a) $p(n_0)$ is true.
  b) For all $k \geq n_0$, $p(k)$ implies $p(k+1)$.

  Desired **Conclusion:** If n is any integer with $n \geq n_0$, then $p(n)$ is true. **If this conclusion is true, induction is a legitimate proof method.**

- **Proof:** Assume a) and b). Let S be the set $\{n \geq n_0 : p(n) \text{ is false}\}$.
  We want to show that S is empty; we do it by contradiction.
  - **Assume that S is non-empty.** Then the well-ordering principle says that S has a smallest element (call it $\mathbf{s}_{min}$).
    We try to show that this leads to a contradiction.
  - Note that $p(\mathbf{s}_{min})$ has to be false. **Why?**
  - $\mathbf{s}_{min}$ cannot be n_0, by hypothesis (a). Thus $\mathbf{s}_{min}$ must be $> n_0$. **Why?**
  - Thus $smin-1 \geq n_0$. Since $\mathbf{s}_{min}$ is the smallest element of S, $\mathbf{s}_{min}$ -1 cannot be an element of S. **What does this say about p ($\mathbf{s}_{min}$ – 1)?**
    - $p(\mathbf{s}_{min} -1)$ is true.
  - By hypothesis (b), using the $k = \mathbf{s}_{min} -1$ case, $p(\mathbf{s}_{min})$ is also true.
    This **contradicts** the previous statement that $p(\mathbf{s}_{min})$ is false.
  - Thus **the assumption that led to this contradiction** (S is nonempty) **must be false**.
  - Therefore S is empty, and $p(n)$ is true for all $n \geq n_0$.

Recap: The Principle of Mathematical Induction

- **To prove that p(n) is true for all n $\geq n_0$:**
  a) Show that $p(n_0)$ is true.

  b) Show that **for all** k $\geq n_0$,

     p(k) implies p(k+1).

     I.e, show that **whenever** p(k) is true, then p(k+1) is true also.

- Let's do it!

Induction example

- To prove: If $n \geq 1$, then $$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

- What is the base case?
- Induction hypothesis?
- What do we need to show in the induction step?
- Procedural matters:
  - Start with one side of the equation/inequality and work toward the other side.
  - DO NOT use the “ write what we want to prove and do the same thing to both sides of the equation to work backwards to what we are assuming" approach.
  - Clearly indicate the step in which you use the induction hypothesis.

Details on next slide

Induction example

- To prove: If n ≥1, then $$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$
- base case: n = 1. Both sides of the equation are equal to 1
- Induction hypothesis? Let k be any number that is ≥ 1. Assume that $\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$
- What do we need to show in the induction step? $\sum_{i=1}^{k+1} i^2 = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6} = \frac{(k+1)(k+2)(2k+3)}{6}$
  - Start with the left side of the equation/inequality and work toward the other side.
  - DO NOT use the " write what we want to prove and do the same thing to both sides of the equation to work backwards to what we are assuming" approach.
  - Clearly indicate the step in which you use the induction hypothesis.

More etails on next slide

Induction Example continued

$\sum_{i=1}^{k+1} i^2 = \sum_{i=1}^{k} i^2 + (k+1)^2$ [property of summation]

$= \frac{k(k+1)(2k+1)}{6} + (k+1)^2$ [induction hypothesis]

$= \frac{k+1}{6}(k(2k+1) + 6(k+1))$ [factor]

$= \frac{k+1}{6}(2k^2 + 7k + 6)$ [Algebra]

$= \frac{k+1}{6}(k+2)(2k+3)$ [this is what we wanted to show]

Note that every step proceeds in the same direction. We never "do the same thing" to both sides of an equation.

Induction is like Recursion

- When we write a recursive program, we make it work for the base case(s).
  - Then, assuming that it works for smaller values, we use the solution for a smaller value to construct a solution for a larger value.
- To prove that a property p(n) is true for all n>0 using (strong) mathematical induction,
  - we show that
    - it is true for the base case (typically n=0 or n=1), and that
    - the truth of p(k) for larger values of k can be derived from the truth of p(j) for all j with $n_0 \leq j < k$.

STRONG INDUCTION

Strong induction

- The following are sufficient to prove that p(n) is true for all $n \geq n_0$
- **(i)** $p(n_0)$ is true.
- **(ii)** for every $k>n_0$, if p(j) is true for all j with $n_0 \leq j < k$, then p(k) is also true.

- Note that we can prove this directly from the Well-Ordering Principle.
  - You may be asked to do that in the homework
  - The proof is almost the same as the proof of ordinary induction.
- Also note that ordinary induction is a special case of strong induction, in which we only assume the truth of p for j=k-1.

Proving Something Using Strong Induction

- **How do we actually construct a proof by strong induction?**
  To show that p(n) is true for all $n \geq n_0$:
  - Step 0: Believe in the "magic."
    - You will show that it's not really magic at all. But you have to believe.
    - If, when you are in the middle of an induction proof, you begin to doubt whether the principle of mathematical induction itself is true, you are sunk!
    - So even if you have some trouble understanding the proof of the principle of mathematical induction, you must believe its truth if you are to be successful in using it to prove things.

Proving Something Using Strong Induction

- **How do we actually construct a proof by strong induction?**
  To show that p(n) is true for all $n \geq n_0$:
- **Step 1 (base case):** Show that $p(n_0)$ is true.
  - Depending on the nature of the induction step (ii), it may also be necessary to show some other base cases as well.
  - For example, an induction proof involving Fibonacci numbers may need two base cases, because the recursive part of the Fibonacci definition expresses F(n) as the sum of two previous values.

Proving Something Using Strong Induction

- **How do we actually construct a proof by induction?**
  To show that p(n) is true for all $n \geq n_0$:
- **Step 2 (induction step)**
  - Let k be any number that is greater than n_0.
    - You can't pick some specific k, you have to do this step for a generic k that is greater than n_0.
  - Assume that p(j) is true for all j that are less than k (and also $\geq n_0$, of course).
  - This is called the induction assumption, and is akin to the assumption that recursive calls to a procedure will work correctly.
  - Then show that p(k) must also be true, using the induction assumption somewhere along the way.

Example

- Every integer n≥1 is a product of zero or more prime integers
- **Proof by strong induction:**
- **Base case.** n=1 is a product of zero prime integers
- **Induction step.** Let *k* be an integer that is greater than 1 The induction assumption is that every positive integer smaller than *k* is a product of prime integers
- We must show that k is a product of prime integers
  - If *k* is prime, then clearly *k* is the product of one prime integer
  - Otherwise *k* is a *composite* integer:
    - i.e., $k = j*m$, where integers *j* and *m* are both greater than one
  - Since *j* and *m* are both larger than 1, $j<k$ and $m<k$
  - Thus by the induction assumption, *m* and *j* are both products of prime integers, and so $k = jm$ is a product of prime integers
- This would be very difficult to prove using ordinary induction

Binary Tree: Recursive definition

- A Binary Tree is either
  - **empty**, or
  - **consists** of:
    - a distinguished node called the root, which contains an element, and
    - A left subtree T_L, which is a binary tree
    - A right subtree T_R, which is a binary tree

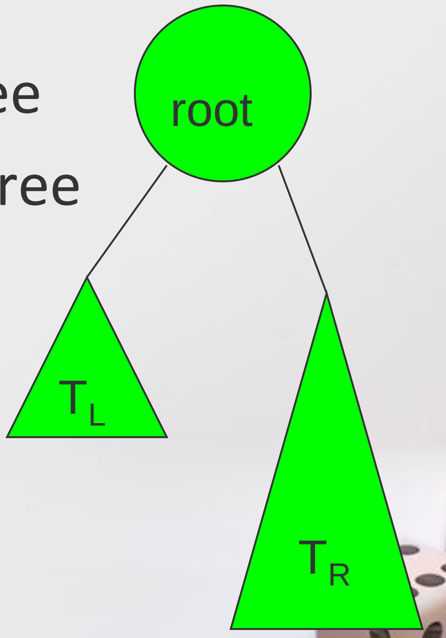
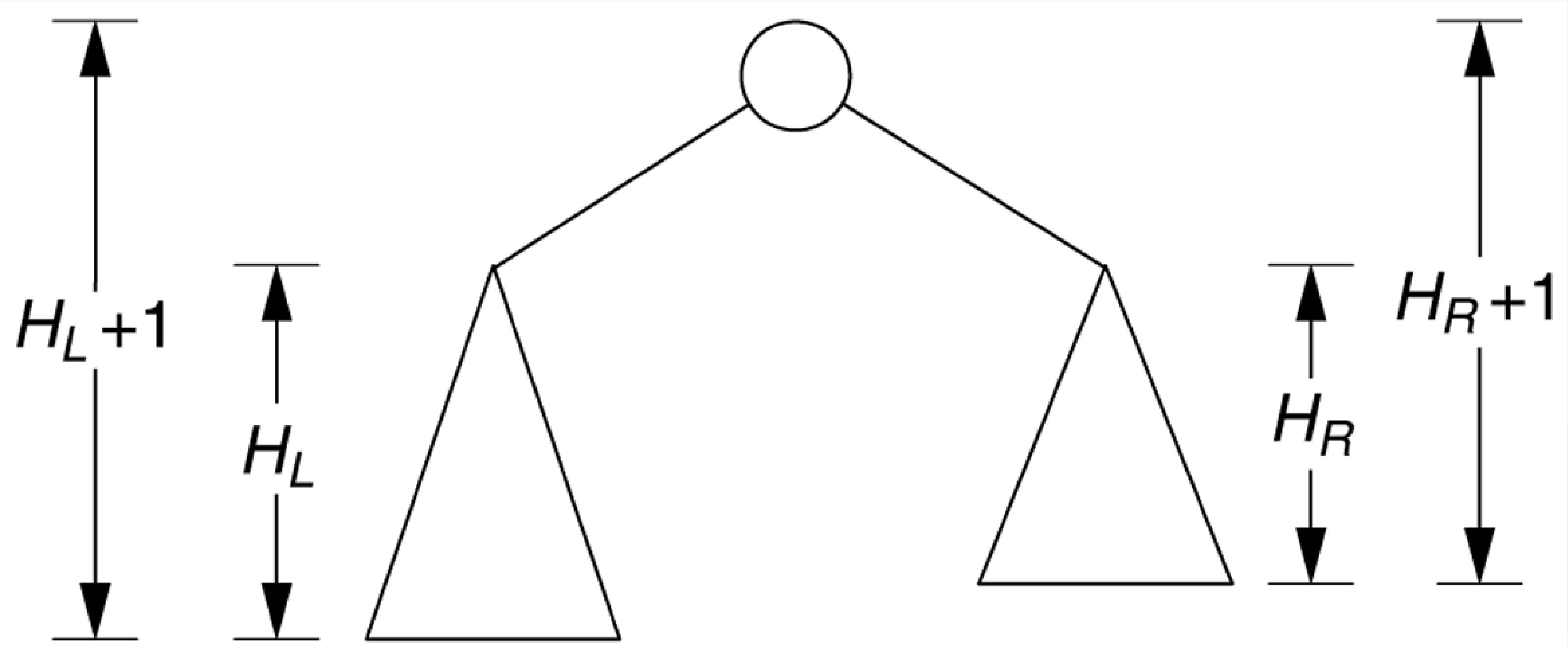


Figure 18.20

Recursive view of the node height calculation:

H_T = Max (H_L + 1, H_R + 1)



Example: a property of binary trees

- Let H(T) be the height of a binary tree T.
- Let N(T) be the size (number of nodes) of T.
- Then $N(T) \leq 2^{H(T)+1}-1$.
- Proof by strong induction:
- **Base case:** H(T) = -1 (tree is empty).
  - Left side of inequality is 0
  - Right side is $2^{-1+1}-1 = 0$
  - Inequality is true.

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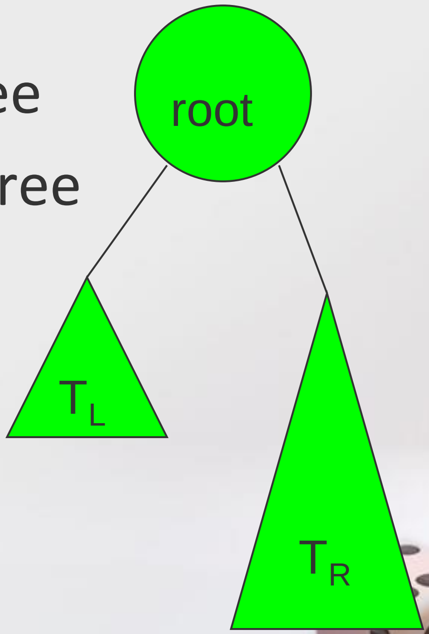

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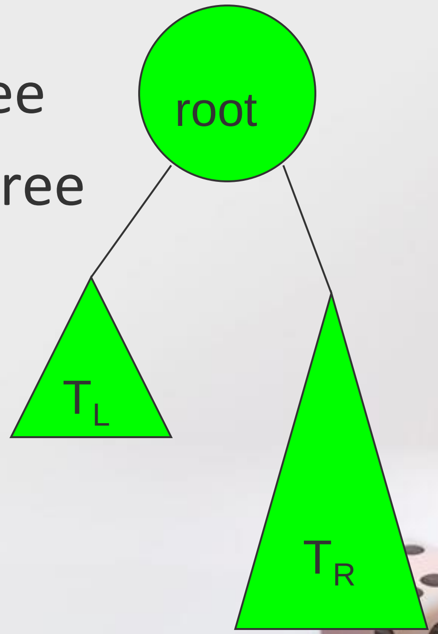

Induction step

Let T be a tree with j > 0 nodes. Assume that the property is true for all smaller trees. In particular, it is true for T's two subtrees. Thus

$N(T) = 1 + N(T_L)+N(T_R)$ [every node is the root or is in a subtree]

$\leq 1 + 2^{h(T_L)+1} - 1 + 2^{h(T_R)+1} - 1$ [induction assumption]

$= 2^{h(T_L)+1} + 2^{h(T_R)+1} - 1$ [rearrange and combine constants]

$\leq 2^{h(T)} + 2^{h(T)} -1$ [height of tallest subtree is one less than height of T]

$= 2(2^{h(T)}) - 1 = 2^1 2^{h(T)} – 1 = 2^{h(T)+1} – 1$ [Algebra]