

Statement of the lemma from the slides:

## MST lemma

- Let  $G$  be a weighted connected graph,
- let  $T$  be any MST of  $G$ ,
- let  $G'$  be any nonempty subgraph of  $T$ , and
- let  $C$  be any connected component of  $G'$ .
- Then:
  - If we add to  $C$  an edge  $e=(v,w)$  that has minimum-weight among all edges that have one vertex in  $C$  and the other vertex not in  $C$ ,
  - $G$  has an MST that contains the union of  $G'$  and  $e$ .

[WLOG,  $v$  is the vertex of  $e$  that is in  $C$ , and  $w$  is not in  $C$ ]

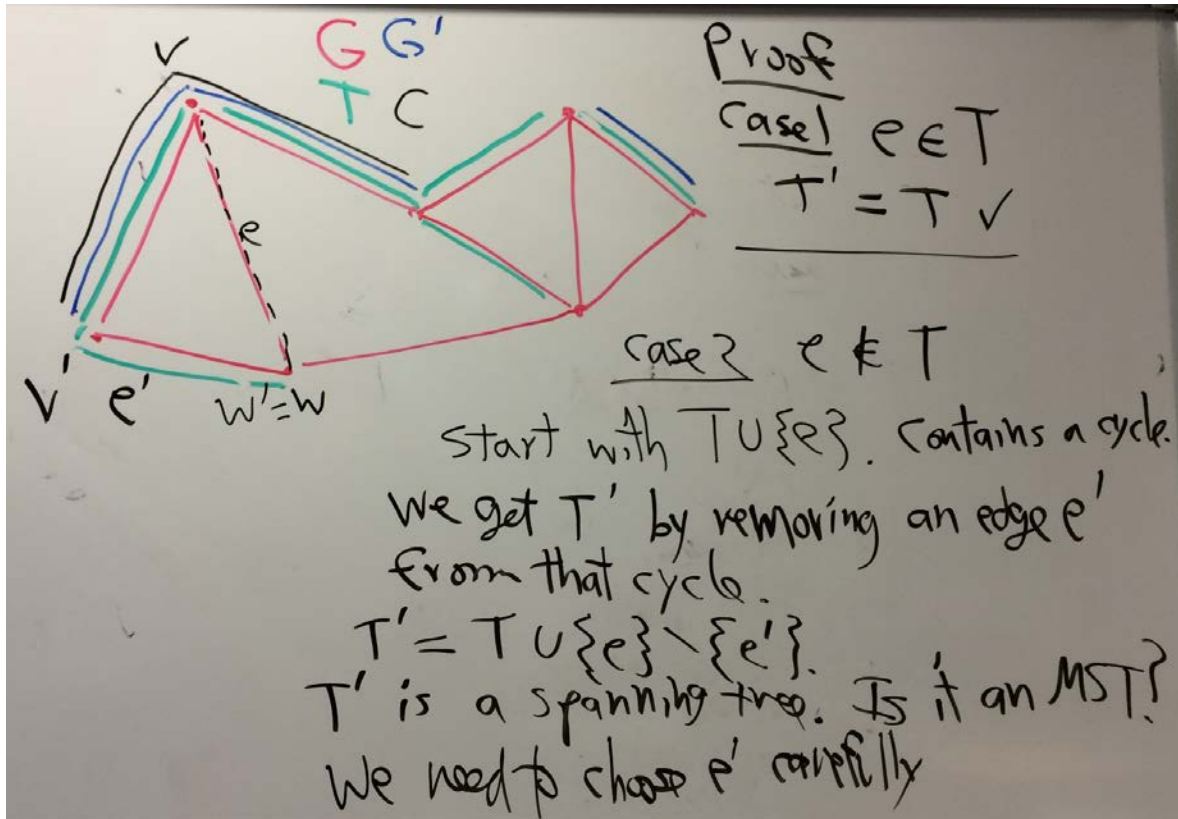
**Summary:** If  $G'$  is a subgraph of an MST, so is  $G' \cup \{e\}$

Notation, hypothesis, and conclusion of the lemma:

$G$  - graph  
 $T$  - MST for  $G$   
 $G'$  - a nonempty subgraph of  $T$ ,  
 $C$  - connected component of  $G'$   
 $e = (v, w)$  an edge with  $v \in C$ ,  $w \in G \setminus C$   
and minimal weight...  
Then  $G' \cup \{e\} \cup \{w\}$  is a subgraph of  
some MST  $T'$

$w$  was left out of the lemma's conclusion on the slides.  $w$  may already be in  $G'$ ; if not, we need to add it when we add a new edge.

**Big picture:** We have a subgraph (call it  $G'$ ) of a MST  $T$  for  $G$ . We want to add another edge in such a way that the union of  $G'$  and this edge is a subgraph of a MST  $T'$  of  $G$ .  $T'$  may or may not be the same MST as  $T$ . But what matters is that if we follow this rule for choosing  $e$ , we will always get a subgraph of a MST. In both Kruskal's and Prim's algorithms, the rule for choosing the next edge satisfies the hypothesis of this lemma, so after each edge choice, we will have a subgraph of a MST



So we show how to choose  $e'$  so that  $T'$  is a MST, then prove that it actually is a MST

About the diagram:

- Note that parallel lines with different colors in the diagram do not imply multiple edges. They imply that a particular edge is part of multiple named subgraphs.
- To keep the graph less complicated, I did not label the edges with their weights.  
 $G'$  is the "subgraph of  $T$ " before choosing the next edge, and  $C$  is a connected component of  $G'$  (we could have let  $C$  be the other connected component of  $G'$  in the diagram)
- $E$  is the "added edge", chosen according to the criterion of the lemma's hypothesis; there are three other edges that would have been candidates for  $e$ ; I assume that the pictured  $e$  was a minimal-weight edge among those 4.

How to choose  $e'$  so that  $T'$  is a minimal spanning tree, then show that it actually is a MST:

In  $T$ , there is a unique path from  $v$  to  $w$ .  
Let  $e'$  be  $(v', w')$ ,  
where  $w'$  is the first node along this path that is not in  $C$ .  
can  $w' = v$ ?  $w' \notin C$   
 $v \in C$

Let  $v'$  be the vertex in the path that immediately precedes  $w$ .

$T$  contains  $e'$  but not  $e$   
 $T$  contains  $e$  but not  $e'$   
only difference

$e$  and  $e'$  are both edges from  $C$  to  $G \setminus C$ .  
Note that  $\text{weight}(e) \leq \text{weight}(e')$   
by criterion for choosing  $e$ .  
 $e'$  is not in  $G'$   
IF  $e' \in G'$ ,  $e' \in C$

$\text{Weight}(T) \geq \text{Weight}(T')$   
 $\text{Weight}(T) \leq \text{Weight}(T')$   
because  $T$  is a MST  
 $\text{Weight}(T) = \text{Weight}(T')$   
 $T'$  is a MST

In the particular graph that I drew,  $w'$  is the same as  $w$ . IN a more complicated graph, this may not be the case.

The above image completes the proof of the lemma.

Now we use the lemma to show that Kruskal's algorithm actually produces a MST for  $G$ .

First we prove by induction:

**Claim:** After every step of Kruskal's algorithm, we have a set of edges that is part of an MST of  $G$

**Proof of claim:** Base case ...

**Induction step:**

- Induction Assumption: before adding an edge we have a subgraph of an MST
- We must show that after adding the next edge we have a subgraph of an MST
- Details:

$G'$

Base case:  $n$  edges ✓

Induction assumption:  
after adding  $i$  edges  
(greedy),  $G'$  is  
subgraph of MST.

Show: still true after  
adding next edge.

satisfies conditions  
of Lemma,  
conclusion of lemma  
is that  $G' \cup \{e\}$   
is subgraph of some MST.

Kruskal correctness:  
After adding  $n-1$   
edges, we have  
subgraph of MST, with  $n-1$  edges  
Must be MST.

Some things I might ask on the final exam:

First, you do not need to memorize the proof. I could give you part or all of the proof and ask you to

- Explain why some of the steps are valid.
- Give you a different diagram with a specific  $G$ ,  $T$ ,  $G'$ ,  $C$ , and  $E$ , and ask you to label other parts, such as  $v$ ,  $w$ ,  $v'$ ,  $w'$ ,  $e'$ ,  $T'$ .

