MA/CSSE 473 Day 23

Review of Binary Heaps and Heapsort

Overview of what you should know about hashing

Answers to student questions

Binary (max) Heap Quick Review

Representation change example

See also Weiss, Chapter 21 (Weiss does min heaps)

- An almost-complete Binary Tree
  - All levels, except possibly the last, are full
  - On the last level all nodes are as far left as possible
  - No parent is smaller than either of its children
  - A great way to represent a **Priority Queue**
- Representing a binary heap as an array:

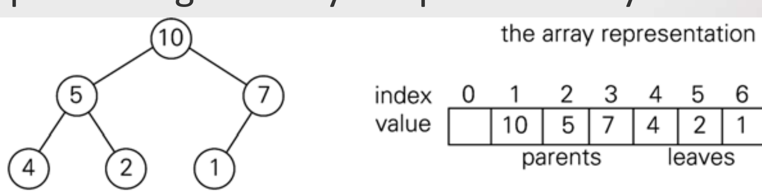


FIGURE 6.10 Heap and its array representation

Insertion and RemoveMax

- Insert an item:
  - Insert at the next position (end of the array) to maintain an almost-complete tree, then "percolate up" within the tree to restore heap property.
- RemoveMax:
  - Move last element of the heap to replace the root, then "percolate down" to restore heap property.
- Both operations are Θ(log n).
- Many more details (done for min-heaps):
  - http://www.rose-hulman.edu/class/csse/csse230/201230/Slides/18-Heaps.pdf

Heap utilitiy functions

```
def percolateDown(a,i, n):
    """Within the n elements of A to be "re-heapified", the two subtrees of A[i]
       are already maxheaps. Repeatedly exchange the element currently in A[i] with
       the largest of its children until the tree whose root is a[i] is a max heap. """
    current = i # root position for subtree we are heapifying
    lastNodeWithChild = n//2  # if a node number is higher than this, it is a leaf.
    while current <= lastNodeWithChild:
        max = current
        if a[max] < a[2*current]:  # if it is larger than its left child.
            max = 2*current
        if 2*current < n and a[max] < a[2*current+1]:  # But if there is a right child,
            max = 2*current + 1                       # right child may be larger than either
        if max == current:
            break  # larger than its children, so we are done.
        swap(a, current, max) # otherwise, exchange, move down tree, and check again.
        current = max

def percolateUp(a,n):
    'Assume that elements 1 through n-1 are a heap; add element n and "re-heapify"'
    # compare to parent and swap until not larger than parent.
    current = n
    while current > 1:  # or until this value is in the root.
        if a[current//2] >= a[current]:
            break
        swap(a, current, current//2)
        current //= 2
```

Code is on-line, linked from the schedule page

HeapSort

- Arrange array into a heap. (details next slide)
- for i = n downto 2:
  a[1]↔a[i], then "reheapify" a[1]..a[i-1]

HeapSort Code

```
# The next two functions tdo the same thing; take an unordered
# array and turn it into a max-heap.  In HW 10, you will show
# that the secondis much more efficient than the first.
# So this first one is not actually called in this code.
def heapifyByInsert(a, n):
    """ Repeatedly insert elements into the heap.
        Worst case number of element exchanges:
            sum of depths of nodes."""
    for i in range(2, n+1):
        percolateUp(a, i)

def buildHeap(a, n):
    """ Each time through the loop, each of node i's two
        subtreees is already a heap.
        Find the correct position to move the root down to
        in order to "reheapify."
        Worst case number of element exchanges:
            sum of heights of nodes."""
    for i in range (n//2, 0, -1):
        percolateDown(a, i, n)

def heapSort(a, n):
    buildHeap(a, n)
    for i in range(n, 1, -1):
        swap(a, 1, i)
        percolateDown(a, 1, i-1)
```

HeapSort: Build Initial Heap

- Two approaches:
  - for i = 2 to n
    percolateUp(i)
  - for j = n/2 downto 1
    percolateDown(j)
- Which is faster, and why?
- What does this say about overall big-theta running time for HeapSort?

Polynomial Evaluation

Problem Reductiion

TRANSFORM AND CONQUER

Horner's Rule

- It involves a representation change.
- Instead of $a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$, which requires a lot of multiplications, we write
- $(\ldots (a_n x + a_{n-1})x + \ldots + a_1)x + a_0$
- code on next slide

Horner's Rule Code

- This is clearly Θ(n).

```
def polyEvalHorner(p, x):
    """ p is a list representing the coefficients.
        p[i] is the coefficient of x^i.
        x is where we are to evaluate p. """
    sum = 0
    for i in range(len(p)-1, -1, -1):
        sum = sum * x + p[i]

    return sum

# evaluate 4x^3 + 3x^2 + 2x + 1 at x=2
print polyEvalHorner([1, 2, 3, 4], 2)
```

Problem Reduction

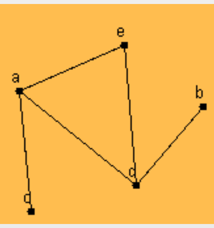
- Express an instance of a problem in terms of an instance of another problem that we already know how to solve.
- There needs to be a one-to-one mapping between problems in the original domain and problems in the new domain.
- **Example:** In quickhull, we reduced the problem of determining whether a point is to the left of a line to the problem of computing a simple 3x3 determinant.
- **Example:** Moldy chocolate problem in HW 9.
  The big question: What problem to reduce it to?
  (You'll answer that one in the homework)

Least Common Multiple

- Let m and n be integers. Find their LCM.
- Factoring is hard.
- But we can reduce the LCM problem to the GCD problem, and then use Euclid's algorithm.
- Note that $\text{lcm}(m,n) \cdot \gcd(m,n) = m \cdot n$
- This makes it easy to find $\text{lcm}(m,n)$

Paths and Adjacency Matrices

- We can count paths from A to B in a graph by looking at powers of the graph's adjacency matrix.



	a	b	c	d	e
a	0	0	1	1	1
b	0	0	0	1	0
c	1	0	0	0	0
d	1	1	0	0	1
e	1	0	0	1	0

	a	b	c	d	e
a	3	1	0	1	1
b	1	1	0	0	1
c	0	0	1	1	1
d	1	0	1	3	1
e	1	1	1	1	2

For this example, I used the applet from http://oneweb.utc.edu/~Christopher-Mawata/petersen2/lesson7.htm, which is no longer accessible

Linear programming

- We want to maximize/minimize a linear function $\sum_{i=1}^{n} c_i x_i$, subject to **constraints**, which are linear equations or inequalities involving the n variables $x_1,\ldots,x_n$.
- The constraints define a region, so we seek to maximize the function within that region.
- If the function has a maximum or minimum in the region it happens at one of the vertices of the convex hull of the region.
- The simplex method is a well-known algorithm for solving linear programming problems. We will not deal with it in this course.
- The Operations Research courses cover linear programming in some detail.

Integer Programming

- A linear programming problem is called an **integer programming** problem if the values of the variables must all be integers.
- The knapsack problem can be reduced to an integer programming problem:
- maximize $\sum_{i=1}^{n} x_i v_i$ subject to the constraints $\sum_{i=1}^{n} x_i w_i < W$ and $x_i \in \{0, 1\}$ for i=1, …, n

Sometimes using a little more space saves a lot of time

SPACE-TIME TRADEOFFS

Space vs time tradeoffs

- Often we can find a faster algorithm if we are willing to use additional space.
- Examples:

Space vs time tradeoffs

- Often we can find a faster algorithm if we are willing to use additional space.
- Give some examples (quiz question)
- Examples:
  - Binary heap *vs* simple sorted array. Uses one extra array position
  - Merge sort
  - Radix sort and Bucket Sort
  - Anagram finder
  - Binary Search Tree (extra space for the pointers)
  - AVL Tree (extra space for the balance code)

Hashing Highlights

- We cover this pretty thoroughly in CSSE 230, and Levitin does a good job of reviewing it concisely, so I'll have you read it on your own (section 7.3).
- On the next slides you'll find a list of things you should know (some of them expressed here as questions)
- Details in Levitin section 7.3 and Weiss chapter 20.
- Outline of what you need to know is on the next slides.
- Will not cover them in great detail in class, since they are typically covered well in 230.

- **Today:** talk with students near you and answer the last two questions on today's handout.

Hashing – You should know, part 1

- Hash table logically contains key-value pairs.
- Represented as an array of size m. H[0..m-1] Typically m is larger than the number of pairs currently in the table.
- Hash function h(K) takes key K to a number in range 0..m
- Hash function goals:
  - Distribute keys as evenly as possible in the table.
  - Easy to compute.
  - Does not require m to be a lot larger than the number of keys in the table.

Hashing – You should know, part 2

- Load factor: ratio of used table slots to total table slots.
  - Smaller → better time efficiency (fewer collisions)
  - Larger → better space efficiency
- Two main approaches to collision resolution
  - Open addressing
  - Se
- Open addressing basic idea
  - When there is a collision during insertion, systematically check later slots (with wraparound) until we find an empty spot.
  - When searching, we systematically move through the array in the same way we did upon insertion until we find the key we are looking for or an empty slot.

Hashing – You should know, part 3

- Open addressing – linear probing
  - When there is a collision, check the next cell, then the next one,…, (with wraparound)
  - Let α be the load factor, and let S and U be the expected number of probes for successful and unsuccessful searches. Expected values for S and U are

α	$\frac{1}{2}(1+\frac{1}{1-\alpha})$	$\frac{1}{2}(1+\frac{1}{(1-\alpha)^2})$
50%	1.5	2.5
75%	2.5	8.5
90%	5.5	50.5

Hashing – You should know, part 4

- Open addressing – double hashing
  - When there is a collision, use another hash function s(K) to decide how much to increment by when searching for an empty location in the table
  - So we look in H(k), H(k) + s(k), H(k) + 2s(k), …, with everything being done mod m.
  - If we we want to utilize all possible array positions, gcd(m, s(k)) must be 1. If m is prime, this will happen.

Hashing – You should know, part 5

- Separate chaining
  - Each of the m positions in the array contains a link ot a structure (perhaps a linked list) that can hold multiple values.
  - Does not have the clustering problem that can come from open addressing.

$$S \approx 1 + \frac{\alpha}{2} \quad \text{and} \quad U = \alpha,$$

  - For more details, including quadratic probing, see Weiss Chapter 20 or my CSSE 230 slides (linked from the schedule page)