MA/CSSE 473 Day 03

Asymptotics

A Closer Look at Arithmetic

With another student, try to write a precise, formal definition of "t(n) is in O(g(n))"

Day 3

- Student questions
  - Course policies?
  - HW assignments?
  - Anything else?
- The two "early course" threads
- Review of asymptotic notation
- Addition and multiplication algorithms

Two threads in lectures

- Each day at the beginning of the course
- A little review (today it's a lot)
- Continue with discussion of efficiency of Fibonacci and arithmetic (if there is time).

Review thread for today:
Asymptotics (O, Θ, Ω)
Mostly a recap of 230 lecture on same topic.
I

Rapid-fire Review: Definitions of O, Θ, Ω

- I will re-use some of my slides from CSSE 230
  - Some of the pictures are from the Weiss book.
- And some of Levitin's pictures
- A very similar presentation appears in Levitin, section 2.2
- Since this is review, we will move much quicker than in 230

Asymptotic Analysis

- We only really care what happens when N (the size of a problem) gets large
- Is the function linear? quadratic? exponential? etc.

Asymptotic order of growth

Informal definitions

A way of comparing functions that ignores constant factors and small input sizes

- O($g(n)$): class of functions $t(n)$ that grow no faster than $g(n)$
- Θ($g(n)$): class of functions $t(n)$ that grow at same rate as $g(n)$
- Ω($g(n)$): class of functions $t(n)$ that grow at least as fast as $g(n)$

Formal Definition

- We will write a precise formal definition of "t(n)∈O(g(n))"
  - This is one thing that students in this course should soon be able to write from memory…
  - … and also understand it, of course!

Big-oh (a.k.a. Big O)

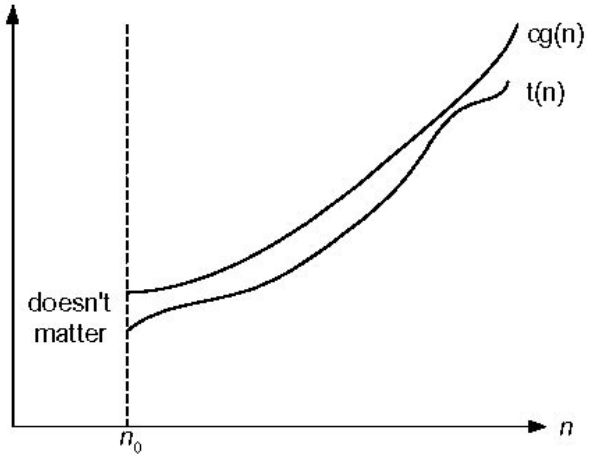


Figure 2.1 Big-oh notation: $t(n) \in O(g(n))$

Prove a Big O Property

- For any function g(n), O(g(n)) is a set of functions
- We say that **t(n) ∈O(g(n))** iff there exist two positive constants **c** and **n_0** such that for all $n \geq n_0$, $t(n) \leq c\ g(n)$
- **Rewrite using ∀ and ∃ notation**
- If f(n)∈O(g(n)) and t(n)∈O(g(n)), then f(n)+t(n)∈O(g(n))
- Let's prove it

Answer (Summer Only)

- Recall: **t(n) ∈O(g(n))** iff there exist two positive constants **c** and **n_0** such that for all $n \geq n_0$, $t(n) \leq c\ g(n)$
- If f(n)∈O(g(n)) and t(n)∈O(g(n)), then f(n)+t(n)∈O(g(n))
- **Proof** By definition, there are constants c_1, c_2, n_1, n_2, such that for all $n \geq n_1$, $f(n) \leq c_1\ g(n)$, and for all $n \geq n_2$, $t(n) \leq c_2\ g(n)$. Let $n_0 = \max(n_1, n2)$, and let $c = c1 + c2$. Then for any $n \geq n_1$, $f(n)+t(n) \leq c_1\ g(n) + c_2\ g(n) = c\ g(n)$.

Big-omega

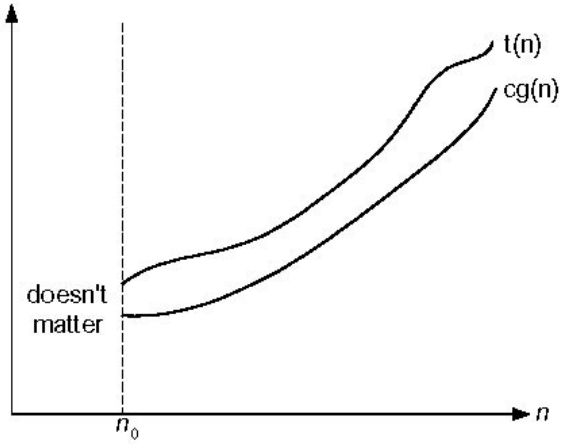


Fig. 2.2 Big-omega notation: $t(n) \in \Omega(g(n))$

Big-theta

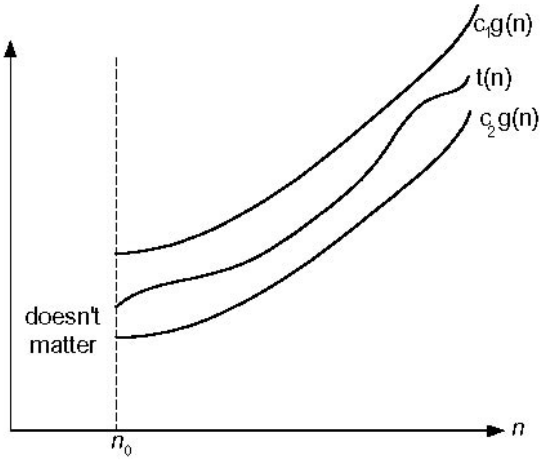


Figure 2.3 Big-theta notation: $t(n) \in \Theta(g(n))$

Big O examples

- **All that we must do to prove that t(n) is O(g(n)) is produce a pair of numbers c and n_0 that work for that case.**
- t(n) = n, g(n) = n^2.
- t(n) = n, g(n) = 3n.
- t(n) = n + 12, g(n) = n.
  We can choose c =3 and n_0 = 6, or c = 4 and n_0 = 4.
- t(n) = n + sin(n)
- t(n) = n^2 + sqrt(n)

In CSSE 230, we do these in great detail in class.

In 473, I say, "work on them if you need review/practice, " and give you a few possible answers on the next slide.

Answers to examples

- For this discussion, assume that all functions have non-negative values, and that we only care about **n**≥0.
  For any function g(n), O(g(n)) is a set of functions We say that a function f(n) is (in) O(g(n)) if there exist two positive constants **c** and $\mathbf{n_0}$ such that for all $n \geq n_0$, $f(n) \leq c\, g(n)$.
- **So all we must do to prove that f(n) is O(g(n)) is produce two such constants.**
- f(n) = n + 12, g(n) = ???.
  - **g(n) = n. Then c =3 and n_0 = 6, or c = 4 and n_0 = 4, etc.**
  - f(n) = n + sin(n): **g(n) = n, c = 2, n_0 = 1**
  - f(n) = n^2 + sqrt(n): **g(n) = n2, c=2, n_0 = 1**

Limits and asymptotics

- Consider the limit $\lim_{n \to \infty} \frac{t(n)}{g(n)}$

- What does it say about asymptotics if this limit is zero, nonzero, infinite?
- We could say that knowing the limit is a sufficient but not necessary condition for recognizing big-oh relationships.
- It will be sufficient for most examples in this course.
- **Challenge:** Use the formal definition of limit and the formal definition of big-oh to prove these properties.

Apply this limit property to the following pairs of functions

1. N and N^2
2. $N^2 + 3N + 2$ and N^2
3. N + sin(N) and N
4. log N and N
5. N log N and N^2
6. N^a and a^N (a >1)
7. a^N and b^N (a < b)
8. $\log_a N$ and $\log_b N$ (a < b)
9. N! and N^N

Big-Oh Style

- **Give tightest bound you can**
  - Saying that **$3N+2 \in O(N^3)$** is true, but not as useful as saying it's O(N) [What about **$\Theta(N^3)$** ?]
- **Simplify:**
  - You *could* say:
  - 3n+2 is O(5n-3log(n) + 17)
  - and it would be technically correct…
  - But **$3n+2 \in O(n)$** is better
- **But… if I ask "true or false: $3n+2 \in O(n^3)$", what's the answer?**
  - True!

BACK TO AND ARITHMETIC THREAD FROM LAST TIME:

The catch!

- Are addition and multiplication constant-time operations?
- We take a closer look at the "basic operations"
- **Addition first:**
- At most, how many digits in the sum of three decimal one-digit numbers?
- Is the same result true in binary and every other base?
- Add two k-bit positive integers (53+35):

Carry:

```
1           1   1   1
    1   1   0   1   0   1     (35)
    1   0   0   0   1   1     (53)
1   0   1   1   0   0   0     (88)
```

- So adding two k-bit integers is O( ).

Multiplication

- Example: multiply 13 by 11

```
            1   1   0   1
          x 1   0   1   1
-----------------------------
            1   1   0   1     (1101 times 1)
        1   1   0   1         (1101 times 1, shifted once)
    0   0   0   0             (1101 times 1, shifted twice)
1   1   0   1                 (1101 times 1, shifted thrice)
-----------------------------
1   0   0   0   1   1   1   1 (binary 143)
```

- There are k rows of 2k bits to add, so we do an O(k) operation k times, thus the whole multiplication is O( ) ?
- Can we do better?

Multiplication by an Ancient Method

- This approach was known to Al Khwarizimi
- According to Dasgupta, *et al*, still used today in some European countries
- Repeat until 1st number is 1, keeping all results:
  - Divide 1st number by 2 (rounding down)
  - double 2nd number
- Example

11	13
5	26
~~2~~	~~52~~
1	104
	143

Then strike out any rows whose first number is even, and add up the remaining numbers in the second column.

- Correct? Analysis

Recursive code for this algorithm

```
def multiply(m, n):
    "multiply two integers m and n, where n >= 0"
    if n == 0:
        return 0
    z = multiply (m, n // 2)
    if n % 2 == 0:
        return 2 * z
    return m + 2 * z

print (multiply(12, 17))
```

For reference: The Master Theorem

- The Master Theorem for Divide and Conquer recurrence relations:
- Consider the recurrence $T(n) = aT(n/b) + f(n)$, $T(1)=c$, where $f(n) = \Theta(n^k)$ and $k \geq 0$,
- The solution is
  - $\Theta(n^k)$ if $a < b^k$
  - $\Theta(n^k \log n)$ if $a = b^k$
  - $\Theta(n^{\log_b a})$ if $a > b^k$

For details, see Levitin pages 483-485 or Weiss section 7.5.3.

Grimaldi's Theorem 10.1 is a special case of the Master Theorem.

We will use this theorem often. You should review its proof soon (Weiss's proof is a bit easier than Levitin's).

New Multiplication Approach

- **Divide and Conquer**
- To multiply two k-bit integers x and y:
  - Split each into its left and right halves so that
    $x = 2^{k/2}x_L + x_R$, and $y = 2^{k/2}y_L + y_R$
  - The straightforward calculation of xy would be
    $(2^{k/2}x_L + x_R)(2^{k/2}y_L + y_R) =$
    $2^k x_L y_L + 2^{k/2}(x_L y_R + x_R y_L) + x_R y_R$
  - Code on next slide
  - We can do the four multiplications of k/2-bit integers using four recursive calls, and the rest of the work (a constant number of bit shifts and additions) in time O(k)
  - Thus T(k) = . Solution?

Code for divide-and-conquer multiplication

```
def multiply(x, y, n):
    """multiply two integers x and y, where n >= 0
       is a power of 2, and as large as the maximum number of bits in x or y"""

    if n == 1:
        return x * y

    n_over_two = n//2

    two_to_the_n_over_two = 1 << n_over_two # a single right bit-shift


    xL, xR = x // two_to_the_n_over_two, x % two_to_the_n_over_two
    yL, yR = y // two_to_the_n_over_two, y % two_to_the_n_over_two
    # note that these two operations could be done by bit shifts and masking.

    p1 = multiply (xL, yL, n_over_two)
    p2 = multiply (xL, yR, n_over_two)
    p3 = multiply (xR, yL, n_over_two)
    p4 = multiply (xR, yR, n_over_two)

    return (p1 << n) + ((p2 + p3) << n_over_two) + p4
```

Can we do better than $O(k^2)$?

- Is there an algorithm for multiplying two k-bit numbers in time that is less than $O(k^2)$?
- **Basis:** A discovery of Carl Gauss (1777-1855)
  - Multiplying complex numbers:
  - **(a + bi)(c+di) = ac – bd + (bc + ad)i**
  - Needs **four** real-number multiplications and **three** additions
  - But **bc + ad = (a+b)(c+d) – ac –bd**
  - And we have already computed **ac** and **bd** when we computed the real part of the product!
  - Thus we can do the original product with 3 multiplications and 5 additions
  - Additions are so much faster than multiplications that we can essentially ignore them.
  - A little savings, but not a big deal until applied recursively!

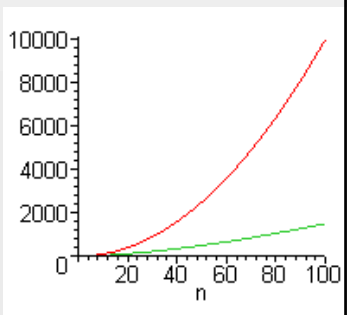
Code for Gauss-based Algorithm

```
def multiply(x, y, n):
    """multiply two integers x and y, where n >= 0
       is a power of 2, and as large as the maximum number of bits in x or y"""

    if n == 1:
        return x * y

    n_over_two = n // 2  # simply shifts the bits one to the right.

    two_to_the_n_over_two = 1 << n_over_two


    xL, xR = x // two_to_the_n_over_two, x % two_to_the_n_over_two
    yL, yR = y // two_to_the_n_over_two, y % two_to_the_n_over_two
    # note that these two operations could be done by bit shifts and masking.

    p1 = multiply (xL,    yL,    n_over_two)
    p2 = multiply (xL+xR, yL+yR, n_over_two)
    p3 = multiply (xR,    yR,    n_over_two)


    return (p1 << n) + ((p2 - p3 - p1) << n_over_two) + p3
```

Is this really a lot faster?

- Standard multiplication: $\Theta(k^2)$
- Divide and conquer with Gauss trick: $\Theta(k^{1.59})$
  - Write and solve the recurrence
- But there is a lot of additional overhead with Gauss, so standard multiplication is faster for small values of n.

```
plot( {n^2, n^1.59}, n=0..100);
```



- In reality we would not let the recursion go down to the single bit level, but only down to the number of bits that our machine can multiply in hardware without overflow.