HW 02 textbook problems and hints

Section 2.1 (Problem 1 – 6 points, 2 - 6)

4. a. *Glove selection* There are 22 gloves in a drawer: 5 pairs of red gloves, 4 pairs of yellow, and 2 pairs of green. You select the gloves in the dark and can check them only after a selection has been made. What is the smallest number of gloves you need to select to have at least one matching pair in the best case? in the worst case?

   b. *Missing socks* Imagine that after washing 5 distinct pairs of socks, you discover that two socks are missing. Of course, you would like to have the largest number of complete pairs remaining. Thus, you are left with 4 complete pairs in the best-case scenario and with 3 complete pairs in the worst case. Assuming that the probability of disappearance for each of the 10 socks is the same, find the probability of the best-case scenario; the probability of the worst-case scenario; the number of pairs you should expect in the average case.

5. a.▷ Prove formula (2.1) for the number of bits in the binary representation of a positive integer.

   b.▷ Prove the alternative formula for the number of bits in the binary representation of a positive integer n:

$$b = \lceil \log_2(n+1) \rceil.$$

   c. What would be the analogous formulas for the number of decimal digits?

   d. Explain why, within the accepted analysis framework, it does not matter whether we use binary or decimal digits in measuring n's size.

HINTS from the textbook (problems 1 and 2):

4. a. Gloves are not socks: they can be right-handed and left-handed.

   b. You have only two qualitatively different outcomes possible. Count the number of ways to get each of the two.

Instructor clarification: A "selection"does not mean choosing a single glove. It means choosing all of the gloves or socks that you are going to choose. You don't look at any of them until you havre chosen all of them.

5. a. First, prove first that if a positive decimal integer n has b digits in its binary representation, then

$$2^{b-1} \leq n < 2^b.$$

Then, take logarithms to base 2 of the terms in this inequality.

b. The proof is similar to the proof of formula (2.1).

c. The formula will be the same, with just one small adjustment to account for the different radix.

d. How can we switch from one logarithm base to another?

Instructor note: Note that there are four parts of this problem

2.2 (3 – 10, 7 – 4 (parts a and d only), 12 -- 6)

3. For each of the following functions, indicate the class $\Theta(g(n))$ the function belongs to. (Use the simplest $g(n)$ possible in your answers.) Prove your assertions.

a. $(n^2 + 1)^{10}$

b. $\sqrt{10n^2 + 7n + 3}$

c. $2n \lg(n+2)^2 + (n+2)^2 \lg \frac{n}{2}$

d. $2^{n+1} + 3^{n-1}$

e. $\lfloor \log_2 n \rfloor$

Instructor note for 2.2.3: For parts a&b, use limits;
for e, use formal definitions of O and Ω; you should find specific values for the c and n_0 in the formulas on pages 53-55.
for c & d, you can use the theorem on p 56.

7. Prove (by using the definitions of the notations involved) or disprove (by giving a specific counterexample) the following assertions.

a. If $t(n) \in O(g(n))$, then $g(n) \in \Omega(t(n))$.

b. $\Theta(\alpha g(n)) = \Theta(g(n))$, where $\alpha > 0$.

c. $\Theta(g(n)) = O(g(n)) \cap \Omega(g(n))$.

d.▷ For any two nonnegative functions $t(n)$ and $g(n)$ defined on the set of nonnegative integers, either $t(n) \in O(g(n))$, or $t(n) \in \Omega(g(n))$, or both.

12. ▷ *Door in a wall* You are facing a wall that stretches infinitely in both directions. There is a door in the wall, but you know neither how far away nor in which direction. You can see the door only when you are right next to it. Design an algorithm that enables you to reach the door by walking at most $O(n)$ steps where n is the (unknown to you) number of steps between your initial position and the door. [Par95]

HINTS from the textbook (problems 3, 4, and 5):

3. Simplify the functions given to single out the terms defining their orders of growth.

7. Prove the correctness of (a), (b), and (c) by using the appropriate definitions; construct a counterexample for (d) (e.g., by constructing two functions behaving differently for odd and even values of their arguments).

12. You should walk intermittently left and right from your initial position until the door is reached.

Instructor note on 2.2.12[2.2.10]: Show that your algorithm is actually O(N).

2.3 (6 – 8, 7 - 10)

2. Find the order of growth of the following sums.

   a. $\sum_{i=0}^{n-1}(i^2+1)^2$ b. $\sum_{i=2}^{n-1}\lg i^2$

   c. $\sum_{i=1}^{n}(i+1)2^{i-1}$ d. $\sum_{i=0}^{n-1}\sum_{j=0}^{i-1}(i+j)$

   Use the $\Theta(g(n))$ notation with the simplest function $g(n)$ possible.

11. Consider the following version of an important algorithm that we will study later in the book.

Algorithm $GE(A[0..n-1, 0..n])$
//Input: An $n \times (n+1)$ matrix $A[0..n-1, 0..n]$ of real numbers
for $i \leftarrow 0$ **to** $n-2$ **do**
    for $j \leftarrow i+1$ **to** $n-1$ **do**
        for $k \leftarrow i$ **to** n **do**
            $A[j,k] \leftarrow A[j,k] - A[i,k] * A[j,i] \;/\; A[i,i]$
return A

a.▷ Find the time efficiency class of this algorithm.

b.▷ What glaring inefficiency does this pseudocode contain and how can it be eliminated to speed the algorithm up?

Instructor note Include a quantitative indication of how much time is gained by removing the glaring inefficiency.

HINTS from the textbook (problems 6 and 7):

2. Find a sum among those in Appendix A that looks similar to the sum in question and try to transform the latter to the former. Note that you do not have to get a closed-end formula for a sum before establishing its order of growth.

11. a. Setting up a sum should pose no difficulties. Using the standard summation formulas and rules will require more effort than in the previous examples, however.

    b. Optimize the algorithm's innermost loop.

Problem 8: (10) 2.4.14 [5.3.10] Celebrity identification

10. *Celebrity problem* A celebrity among a group of n people is a person who knows nobody but is known by everybody else. The task is to identify a celebrity by only asking questions to people of the form: "Do you know him/her?" Design an efficient algorithm to identify a celebrity or determine that the group has no such person. How many questions does your algorithm need in the worst case?

HINTS from the textbook (problem 8):

10. Solve first a simpler version in which a celebrity must be present.

Instructor clarification:

be sure to start this one early! "Efficient" in this case means making the number of questions for N people as small as you can, and you should say how many questions are required by your approach in the worst case.

The remaining problems are not from the textbook; their statements are directly in the assignment document.

9. Master Theorem Proof (8 points).
   These questions refer to the proof in Weiss section 7.5.3 (available on ANGEL).

   (a) (5 points) (7.11) Why is it $O(A^M)$? Why is the second equation true ($O(A^M) = O(N^{\log_B A})$.)?
   (b) (3 points) Sentence after (7.11). Why does the sum contain that many terms? Why does $A = B^k$ imply $A^M = N^k$?

10. Dasgupta questions (6 points, 2 for each part)

(a) What does Dasgupta say are the two main ideas that changed the world? Do you agree? What else might you include in the list?
(b)Why is the simple algorithm at the bottom of page 4 actually not O(n)?
(c) Show how to use al-Khwarizmi's technique to multiply 9 (first column) by 15 (second column).

Note on problem 10. These are easy "did you read this?" problems. On later assignments there will be harder problems based on the Dasgupta excerpt.