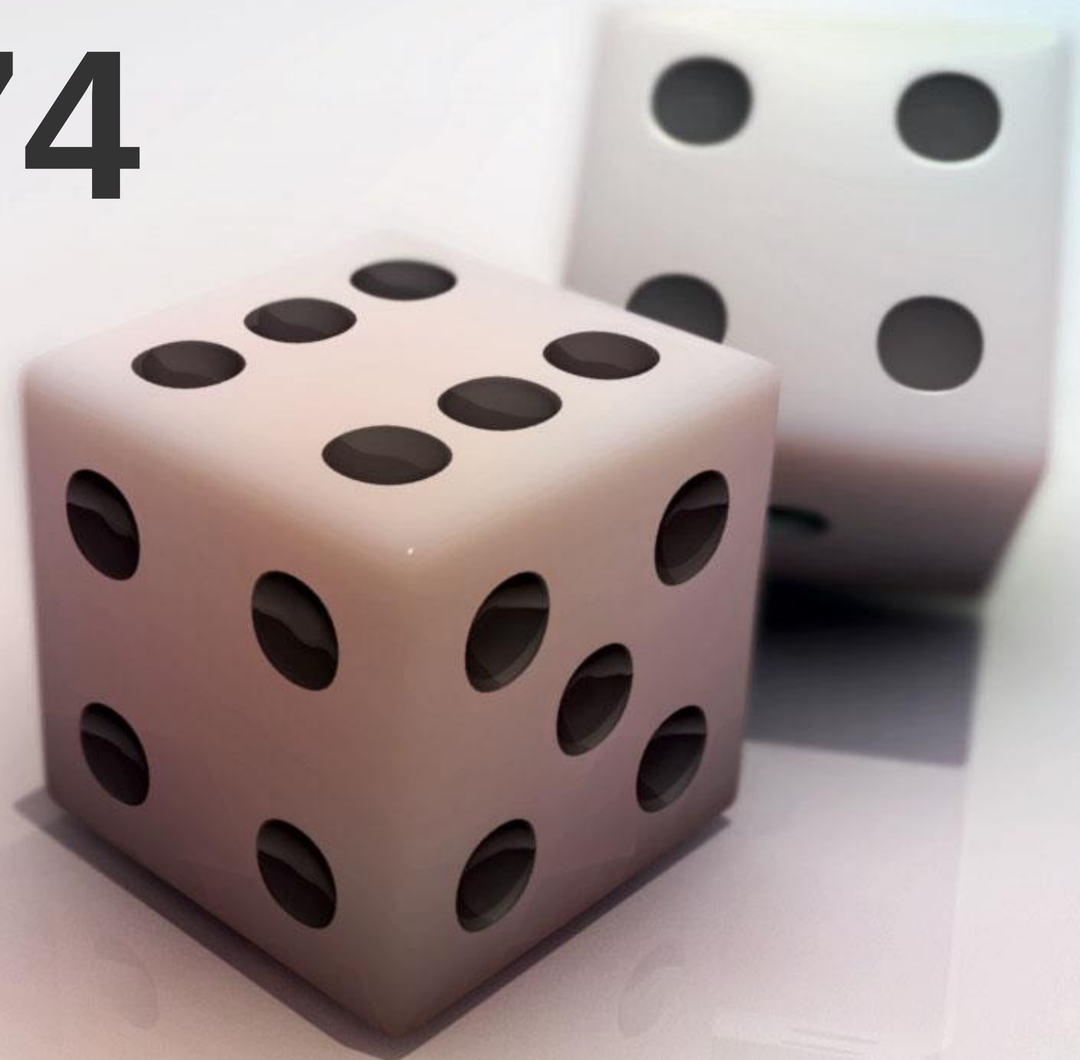
MA/CSSE 473/474

How (not) to do an induction proof

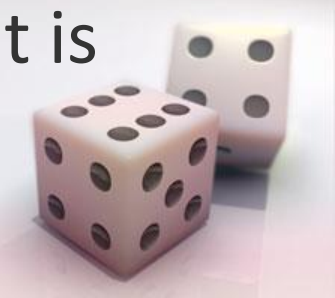


Reminder of some simple logic

- A →B (A implies B) means that whenever A is true, B is true also. The only way A → B can be false is when A is *true* and B is *false*.
- The inverse, B → A, is ***not*** the same as A → B.
- Example. Let a be x<5 and B be x<12
  - Clearly, A →B is true for all x, because every number that is less than 5 is also less then 12.
  - But B → A is not always true, e.g., x = 8.
- If you are trying to prove A →B, it is **incorrect** to instead prove B → A.

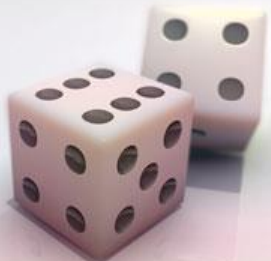
B	A	A-->B	B-->A
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

- It is the ***contrapositive***, not the ***inverse***, that is equivalent to the original:
  - i.e., A →B if and only if (not B) → (not A).



An induction proof is not like solving an equation!

- Sometimes solving an equation can lead to the insight of how we might prove something.
- But the induction process is more like *checking* the solution that you have already found.
- The "do the same thing to both sides of the equation" approach is seldom appropriate as part of an induction proof.



Induction Review

- To show that property P(n) is true for all integers $n \geq n_0$, it suffices to show:
  - **Ordinary Induction**
    - $P(n_0)$ is true
    - For all $k \geq n_0$, if P (k) is true, then P(k+1) is also true.

  or

  - **Strong Induction**
    - $P(n_0)$ is true (sometimes you need multiple base cases)
    - For all $k > n_0$, if P(j) is true for all j with $n_0 \leq j < k$, then P(k) is also true.

In this context, a **property** is a function whose domain is a subset of the non-negative integers and whose range is {true, false}

Induction Review

- To show that property P(n) is true for all integers $n \geq n_0$, it suffices to show:
  - **Ordinary Induction**
    - $P(n_0)$ is true
    - For all $k \geq n_0$, if P (k) is true, then P(k+1) is also true.

Note that what we need to prove for all k is

$$P(k) \rightarrow P(k+1), \text{ not } P(k+1) \rightarrow P(k).$$

Thus it is **incorrect** to instead start with P(k+1) and show that the induction assumption leads to P(k) , or that it leads to some true statement.

You will lose points on homework assignments if you use the "start with P(k+1) and work backwards" approach.

Induction example

- For all N≥0, $$\sum_{i=1}^{N} i \cdot 2^{i} = 2^{N+1}(N-1)+2$$
  - This is formula 7 from the "Important Summation Formulas" in Appendix A of the 473 book, and is useful in one of the HW2 problems.
  - The idea applies ot 474 and 230 students as well.

Induction example

- For all N≥0, $$\sum_{i=1}^{N} i \cdot 2^{i} = 2^{N+1}(N-1)+2$$
  - This is formula 7 on P 470, and is useful in one of the HW2 problems.
- **Proof of base case**, N=0.
  - Left side is 0 because the sum is empty.
  - Right side is 2 – 2 = 0.
- **Induction step:** next slide

Induction example, continued

- **What we need to show.**
- For all N≥0, if $\sum_{i=1}^{N} i \cdot 2^i = 2^{N+1}(N-1)+2$, then

$$\sum_{i=1}^{N+1} i \cdot 2^i = 2^{N+2}(N)+2$$

Next, an invalid "proof":

$$\sum_{i=1}^{N+1} i \cdot 2^i = 2^{N+2}(N)+2$$

$$(N+1)2^{N+1} + \sum_{i=1}^{N} i \cdot 2^i = 2^{N+2}(N)+2$$

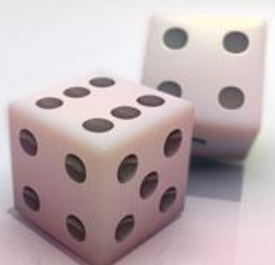
$$(N+1)2^{N+1} + 2^{N+1}(N-1)+2 = 2^{N+2}(N)+2$$

$$(N+1)2^{N+1} + 2^{N+1}(N-1) = 2^{N+2}(N)$$

$$2^{N+1}(N+1+N-1) = 2^{N+2}(N)$$

$$2 * 2^{N+1}(N) = 2^{N+2}(N)$$

$$2^{N+2}(N) = 2^{N+2}(N)$$



Correct induction step

- **What we need to show.**
- For all N≥0, if $\sum_{i=1}^{N} i \cdot 2^i = 2^{N+1}(N-1)+2$, then

$$\sum_{i=1}^{N} i \cdot 2^i = 2^{N+1}(N-1)+2$$

Now, a correct proof:

$$\sum_{i=1}^{N+1} i \cdot 2^i = (N+1)2^{N+1} + \sum_{i=1}^{N} i \cdot 2^i$$

$$= (N+1)2^{N+1} + 2^{N+1}(N-1)+2$$

$$= 2^{N+1}(N+1+N-1)+2$$

$$= 2 * 2^{N+1}(N)+2$$

$$= 2^{N+2}(N)+2$$