473 Levitin problems and hints HW 06

Problem 1: **(6)** 3.5.3 [5.2.3] Independence of properties from specific DFS traversals. Explain your answers.

3. Let G be a graph with n vertices and m edges.

   a. True or false: All its DFS forests (for traversals starting at different vertices) will have the same number of trees?

   b. True or false: All its DFS forests will have the same number of tree edges and the same number of back edges?

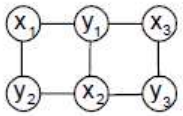
Author's Hints:

3. a. What is the number of such trees equal to?

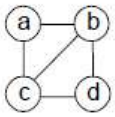
   b. Answer this question for connected graphs first.

Problem 2: **(10)** 3.5.8a [5.2.8a] Bipartite graph checking using DFS

8. A graph is said to be ***bipartite*** if all its vertices can be partitioned into two disjoint subsets X and Y so that every edge connects a vertex in X with a vertex in Y. (We can also say that a graph is bipartite if its vertices can be colored in two colors so that every edge has its vertices colored in different colors; such graphs are also called ***2-colorable***). For example, graph (i) is bipartite while graph (ii) is not.



(i)



(ii)

   a. Design a DFS-based algorithm for checking whether a graph is bipartite.

   b. Design a BFS-based algorithm for checking whether a graph is bipartite.

Author's Hints:

8. Use a DFS forest and a BFS forest for parts (a) and (b), respectively.

Problem 3: **(5)** 4.1.1 [5.1.1] Ferrying Soldiers

1. *Ferrying soldiers* A detachment of n soldiers must cross a wide and deep river with no bridge in sight. They notice two 12-year-old boys playing in a rowboat by the shore. The boat is so tiny, however, that it can only hold two boys or one soldier. How can the soldiers get across the river and leave the boys in joint possession of the boat? How many times need the boat pass from shore to shore?

Author's Hints:

1. Solve the problem for $n = 1$.

Problem 4: **(5)** 4.1.4 [5.1.3] generate power set

4. Design a decrease-by-one algorithm for generating the power set of a set of n elements. (The power set of a set S is the set of all the subsets of S, including the empty set and S itself.)

Author's Hints:

4. Use the fact that all the subsets of an n-element set $S = \{a_1, \ldots, a_n\}$ can be divided into two groups: those that contain a_n and those that do not.

Problem 5: **(5)** (not in book) [5.1.9] binary insertion sort efficiency

9. ▷ Binary insertion sort uses binary search to find an appropriate position to insert $A[i]$ among the previously sorted $A[0] \leq \ldots \leq A[i-1]$. Determine the worst-case efficiency class of this algorithm.

Author's Hints:

9. The order of growth of the worst-case number of key comparisons made by binary insertion sort can be obtained from formulas in Section 4.3 and Appendix A. For this algorithm, however, a key comparison is not the operation that determines the algorithm's efficiency class. Which operation does?

Problems 6: **(9)** 4.2.6 [5.3.6] dag source

In the 3rd edition, it says "Prove that a **nonempty** dag must have at least one source." That additional word is necessary!

6. a. Prove that a dag must have at least one source.

b. How would you find a source (or determine that such a vertex does not exist) in a digraph represented by its adjacency matrix? What is the time efficiency of this operation?

c. How would you find a source (or determine that such a vertex does not exist) in a digraph represented by its adjacency lists? What is the time efficiency of this operation?

Author's Hints:

6. a. Use a proof by contradiction.

b. If you have difficulty answering the question, consider an example of a digraph with a vertex with no incoming edges and write down its adjacency matrix.

c. The answer follows from the definitions of the source and adjacency lists.

Problem 7: **(9)** 4.2.9 [5.3.9] Strongly connected components of a digraph

9. A digraph is called ***strongly connected*** if for any pair of two distinct vertices u and v, there exists a directed path from u to v and a directed path from v to u. In general, a digraph's vertices can be partitioned into disjoint maximal subsets of vertices that are mutually accessible via directed paths of the digraph; these subsets are called ***strongly connected components***. There are two DFS-based algorithms for identifying strongly connected components. Here is the simpler (but somewhat less efficient) one of the two:

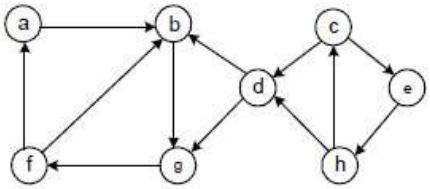
Step 1. Do a DFS traversal of the digraph given and number its vertices in the order that they become dead ends.

Step 2. Reverse the directions of all the edges of the digraph.

Step 3. Do a DFS traversal of the new digraph by starting (and, if necessary, restarting) the traversal at the highest numbered vertex among still unvisited vertices.

The strongly connected components are exactly the subsets of vertices in each DFS tree obtained during the last traversal.

a. Apply this algorithm to the following digraph to determine its strongly connected components.



b. What is the time efficiency class of this algorithm? Give separate answers for the adjacency matrix representation and adjacency list representation of an input graph.

c. How many strongly connected components does a dag have?

Author's Hints:

9. a. Trace the algorithm on the input given by following the steps of the algorithm as indicated.

b. Determine the efficiency for each of the three principal steps of the algorithm and then determine the overall efficiency. Of course, the answers depend on whether a digraph is represented by its adjacency matrix or by its adjacency lists.

Problem 8: **(15)** Not from the textbook.

(Miller-Rabin test) For this problem you will need the excerpt from the Dasgupta book that is posted on Moodle.
Let N = 1729 (happens to be a Carmichael number, but you should not assume that as you discover the answers) for all parts of this problem.

(a) How many values of **a** in the range 1..1728 pass the Fermat test [i.e. $a^{1728} \equiv 1 \pmod{1729}$]?
(b) For how many of these "Fermat test positive" values of **a** from part (a) does the Miller-Rabin test provide a witness that N is actually composite?
(c) If we pick **a** at random from among 1, 2,..,N, what is the probability that running the Miller-Rabin test on **a** will show that N is composite? Rabin showed that for any N, the probability is at least 75%; what is the probability for the N=1729 case?

[**Hint**: writing some code is likely help you in this problem. If you do that, include the code in your submission].

Problem 9: **(9)** 4.3.2 [5.4.2] Examples of permutation generation algorithms

2. Generate all permutations of $\{1, 2, 3, 4\}$ by

   a. the bottom-up minimal-change algorithm.

   b. the Johnson-Trotter algorithm.

   c. the lexicographic–order algorithm.

You do not have to write any code, but you can do it that way if you wish.

Author's Hints:

2. We traced the algorithms on smaller instances in the section.

Problem 10: **(10)** 4.3.10 [5.4.10] Generation of all k-combinations from an n-element set

10. ► Design a decrease-and-conquer algorithm for generating all combinations of k items chosen from n, i.e., all k-element subsets of a given n-element set. Is your algorithm a minimal-change algorithm?

Author's Hints:

10. There are several decrease-and–conquer algorithms for this problem. They are more subtle than one might expect. Generating combinations in a predefined order (increasing, decreasing, lexicographic) helps with both a design and a correctness proof. The following simple property is very helpful. Assuming with no loss of generality that the underlying set is $\{1, 2, ..., n\}$, there are $\binom{n-i}{k-1}$ k-subsets whose smallest element is i, $i = 1, 2, ..., n-k+1$.

Problem 11: **(10)** 4.3.11 [5.4.11]

11. *Gray code and the Tower of Hanoi*

    (a) ▷ Show that the disk moves made in the classic recursive algorithm for the Tower-of-Hanoi puzzle can be used for generating the binary reflected Gray code.

    (b) ► Show how the binary reflected Gray code can be used for solving the Tower-of-Hanoi puzzle.

Author's Hints:

11. Represent the disk movements by flipping bits in a binary n-tuple.

Problem 12: **(10)** 4.3.12 Not in 2nd edition

12. *Fair attraction* In olden days, one could encounter the following attraction at a fair. A light bulb was connected to several switches in such a way that it lighted up only when all the switches were closed. Each switch was controlled by a push button; pressing the button toggled the switch, but there was no way to know the state of the switch. The object was to turn the light bulb on. Design an algorithm to turn on the light bulb with the minimum number of button pushes needed in the worst case for n switches.

Author's Hints:

12. Thinking about the switches as bits of a bit string could be helpful but not necessary.