Decrease by a constant factor

Decrease by a variable amount

A FEW LEFTOVER DECREASE-AND-CONQUER ALGORITHMS

Decrease by a Constant Factor

- **Examples that we have already seen:**
  - Binary Search
  - Exponentiation (ordinary and modular) by repeated squaring
  - Multiplication à la Russe (The Dasgupta book that I often used for the first part of the course calls it "European" instead of "Russian")
    - Example

11	13
5	26
~~2~~	~~52~~
1	104
	143

Then strike out any rows whose first number is even, and add up the remaining numbers in the second column.

Fake Coin Problem

- We have n coins
- All but one have the same weight
- One is lighter
- We have a balance scale with two pans.
- All it will tell us is whether the two sides have equal weight, or which side is heavier
- What is the minimum number of weighings that will guarantee that we find the fake coin?
- Decrease by factor of two?

Decrease by a variable amount

- Search in a Binary Search Tree
- Interpolation Search
  - See Levitin, pp190-191
  - Also Weiss, Section 5.6.3

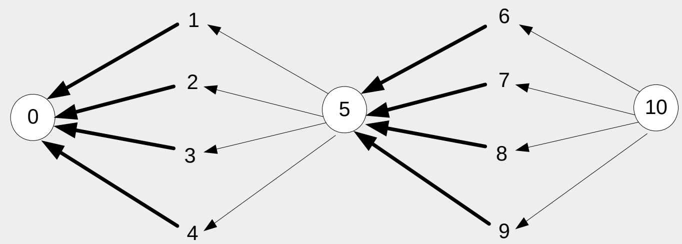
Median finding

- Find the k^{th} element of an (unordered) list of n elements
- Start with quicksort's partition method
- Informal analysis

One Pile Nim

- There is a pile of n chips. Two players take turns by removing from the pile at least 1 and at most m chips. (The number of chips taken can vary from move to move.)
- The winner is the player that takes the last chip.
- Who wins the game – the player moving first or second, if both players make the best moves possible?
- It's a good idea to analyze this and similar games "backwards", i.e., starting with n = 0, 1, 2, …

Graph of One-Pile Nim with *m* = 4



- Vertex numbers indicate n, the number of chips in the pile. The losing position for the player to move are circled. Only winning moves from a winning position are shown.
- **Generalization:** The player moving first wins iff n is not a multiple of 5 (more generally, m+1);
  - The winning move is to take n mod 5 (n mod (m+1)) chips on every move.

Multi-Pile Nim

- There are multiple piles of chips. Two players take turns by removing from any single pile at least 1 and at most all of that pile's chips. (The number of chips taken can vary from move to move)
- The winner is the player that takes the last chip.
- What is the winning strategy for 2-pile Nim?
- For the general case, consider the "Nim sum", $x \oplus y$, which is the integer obtained by bitwise XOR of corresponding bits of two non-negative integers x and y. What is $6 \oplus 3$?

Multi-Pile Nim Strategy

- Solution by C.L. Bouton:
- The first player has a winning strategy iff the nim sum of the "pile counts" is not zero.
- **Let's prove it.** Note that $\oplus$ is commutative and associative.
- Also note that for any non-negative integer k, $k \oplus k$ is zero.

Multi-Pile Nim Proof

- **Notation:**
  - Let $x_1, \ldots, x_n$ be the sizes of the piles before a move, and $y_1, \ldots, y_n$ be the sizes of the piles after that move.
  - Let $s = x_1 \oplus \ldots \oplus x_n$, and $t = y_1 \oplus \ldots \oplus y_n$.
- **Observe:** If the chips were removed from pile k, then $x_i = y_i$ for all $i \neq k$, and $x_k > y_k$.
- **Lemma 1:** $t = s \oplus x_k \oplus y_k$.
- **Lemma 2:** If $s = 0$, then $t \neq 0$.
- **Lemma 3:** If $s \neq 0$, it is possible to make a move such that $t=0$. [after proof, do an example].
- Proof of the strategy is then a simple induction.
  (do this one with a partner.
  Great HW or exam question.)

Josephus problem - background

- Flavius Josephus was a Jewish general and historian who lived and wrote in the 1st century AD
- Much of what we know about 1st century life in Israel (and the beginnings of Christianity) before and after the Roman destruction of the Jewish temple in 70 AD comes from his writings
- The "Josephus problem" is based on an odd suicide pact that he describes
  - He and his men stood in a circle and counted off
  - Every other person (or every third person, accounts vary) was killed
  - The last person was supposed to kill himself
  - He must have been the next-to-last person!
  - When it got down to two people, he persuaded the other person that they should surrender instead
- http://en.wikipedia.org/wiki/Josephus