CSSE 461 – Computer Vision
Rose-Hulman Institute of Technology
Computer Science and Software Engineering Department

Linear Algebra Review/Primer

1 Introduction

The goal of this review/primer is to refresh your memory of and/or expose you to some linear algebra concepts that will be needed as we move through the course. I will highlight some of the connections between linear algebra and computer vision and provide a number of relevant excercises, but I will not present in detail fundamental linear algebra concepts. For detailed information on these concepts see your linear algebra book or 18.06 Linear Algebra the MIT opencourseware.

2 Notation

$\mathbb{R}$ denotes the real numbers and $\mathbb{R}^n$ denotes an $n-$tuple of real numbers. For example, $[x, y]$ is an element of $\mathbb{R}^2$.

Vectors (or points) in $\mathbb{P}^2$ are denoted by bold face lower case letters (e.g. $\mathbf{x}$). Vectors will by default be column vectors. Row vectors are denoted by the transpose of a column vector (e.g. $\mathbf{x}^\top$). Vectors (or points) in $\mathbb{P}^3$ are denoted by bold face upper case letters (e.g. $\mathbf{X}$).

Matrices are denoted by bold face upper case letters (e.g. $\mathbf{H}$). Matrices of a specific size are denoted by bold face upper case letters with a subscript indicating the size. For example, $\mathbf{H}_2$ indicates a 2×2 matrix and $\mathbf{H}_{2\times3}$ indicates a 2×3 matrix.

Planes are denoted by a bold face lower case pi (e.g. π).

3 General Matrix Properties

Composing Matrices: For the following expressions, what is the size of the resulting matrix if it exists.

1. $\mathbf{M}_{1\times3}\mathbf{N}_{3\times1}$
2. $\mathbf{N}_{3\times1}\mathbf{M}_{1\times3}$
3. $\mathbf{N}_{3\times2}\mathbf{M}_{1\times3}$
4. $\mathbf{M}_{1\times3}\mathbf{N}_{3\times2}$

Types of matrices: Describe the following types of matrices:

1. Orthogonal
2. Orthonormal
3. Diagonal
4. Symmetric
5. Skew Symmetric

Note: An orthonormal matrix is also known as a rotation matrix.

Interpreting Matrices

1. Let $\mathbf{R}_{3\times3}$ be a rotation matrix. Show that the matrix can be written as:

$$\begin{bmatrix} \mathbf{r}_{\mathrm{x}}^{\top} \\ \mathbf{r}_{\mathrm{y}}^{\top} \\ \mathbf{r}_{\mathrm{z}}^{\top} \end{bmatrix}$$

   Where $\mathbf{r}_i$ is a vector, expressed in the global coordinate frame, defining the i^{th} axis in the transformed coordinate frame.

2. A conic can be expressed in algebraic form as:

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

   The connic can also be expressed in matrix form as $\mathbf{x}^{\top}\mathbf{C}\mathbf{x} = 0$, where $\mathbf{x}$ is a point in $\mathbb{P}^2$ and $\mathbf{C}$ is a 3×3 matrix. Find the elements of $\mathbf{C}$ in terms of a, b, c, d, e and f.

3. The crossproduct of two 3-vectors (e.g. $\mathbf{x} \times \mathbf{y}$ can be represented as the product of a matrix and a vector. Specifically:

$$[\mathbf{x}]_{\times}\,\mathbf{y}$$

   where $[\mathbf{x}]_{\times}$ is a 3×3 matrix. Find the coefficients of $[\mathbf{x}]_{\times}$.

The Rank of a Matrix

1. Define the rank of a matrix.
2. What is the maximum rank of an $m \times n$ matrix?
3. If an $n \times n$ matrix is singular what is its maximum rank?

The Determinant of a Matrix

1. What is the determinant of a matrix?
2. How is the determinant calculated for a 2×2 matrix?
3. How is the determinant calculated for a 3×3 matrix?
4. What is the determinant of a singular matrix?

The Transpose of a Matrix

1. Define the transpose of a matrix.
2. What is the transpose of:
   (a) $\mathbf{R}$, an orthonormal matrix
   (b) $\mathbf{D}$, a diagonal matrix
   (c) $\mathbf{S}$, a symmetric matrix
   (d) $\mathbf{A}$, a skew symmetric matrix

   Express the result in its simplest form.
3. Simplify $(\mathbf{MN})^\top$.

The Inverse of a Matrix

1. Define the inverse of a matrix.
2. What is the inverse of:
   (a) $\mathbf{R}$, an orthonormal matrix
   (b) $\mathbf{D}$, a diagonal matrix

   Express the result in its simplest form.
3. Simplify $(\mathbf{MN})^{-1}$.

The Null Space of a Matrix

1. Define the null space of a matrix.
2. What is the left null space of a matrix?
3. If a matrix is full rank what is its null space?
4. If matrix $\mathbf{M}_{4\times 4}$ has a rank of 2 what is the dimensionality of the null space?
5. How does the null space of a matrix relate to the solutions of the equations $\mathbf{Ax} = \mathbf{0}$ and $\mathbf{Ax} = \mathbf{b}$.

The Condition Number of a Matrix

1. What is the condition number of a matrix?
2. What is preconditioning of a matrix?
3. What is data normalization?

Solving systems of linear equations

1. Explain how the following can be used to solve non-homogeneous equations (e.g. $\mathbf{Ax} = \mathbf{b}$):
   (a) Matrix inverse
   (b) Psuedo inverse
   (c) Gaussian elimination
   (d) Factorization (e.g. LU, QR, etc.)

   Does whether the system is exactly determined or overdetermined change which strategies will work?
2. Explain how the following can be used to solve homogeneous equations (e.g. $\mathbf{Ax} = \mathbf{0}$):
   (a) The null space
   (b) SVD (singular value decomposition)

   Does whether the system is exactly determined or overdetermined change which strategies will work?

Singular Value Decomposition

1. Describe the singular value decomposition.
2. What is the significance of the columns of U?
3. What is the significance of the elements of D?
4. What is the significance of the columns of V?