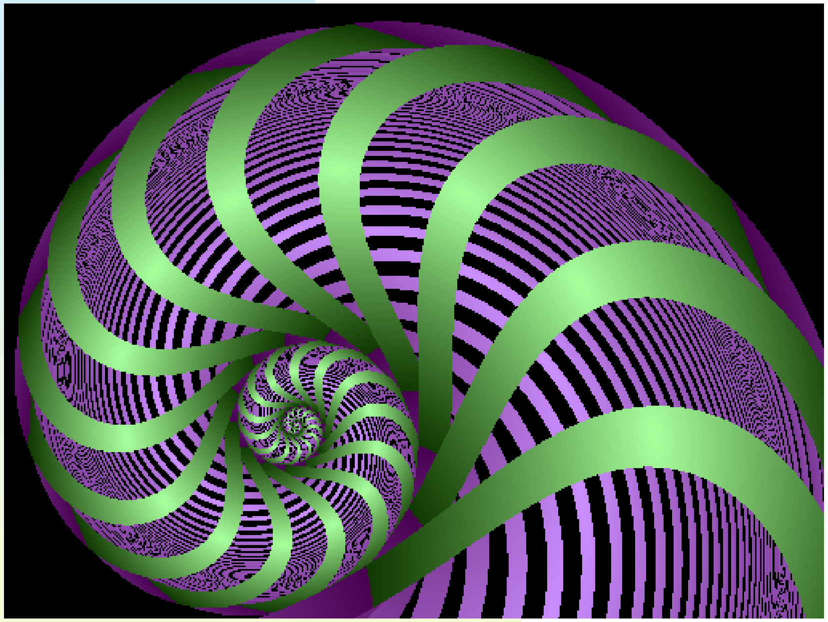
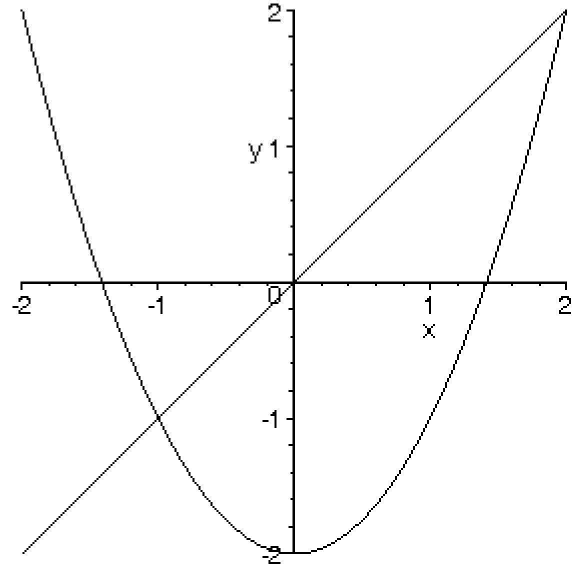
Session overview



- A closer study of the quadratic iterator, $Q_c(x) = x^2 + c$
- Project 5 due now to repository
- Project 6: Independent Demo/Exploration
- Stretch and fold:
- http://www.boingboing.net/2008/04/10/howto-make-fractal-c.html

The case $c = -2$

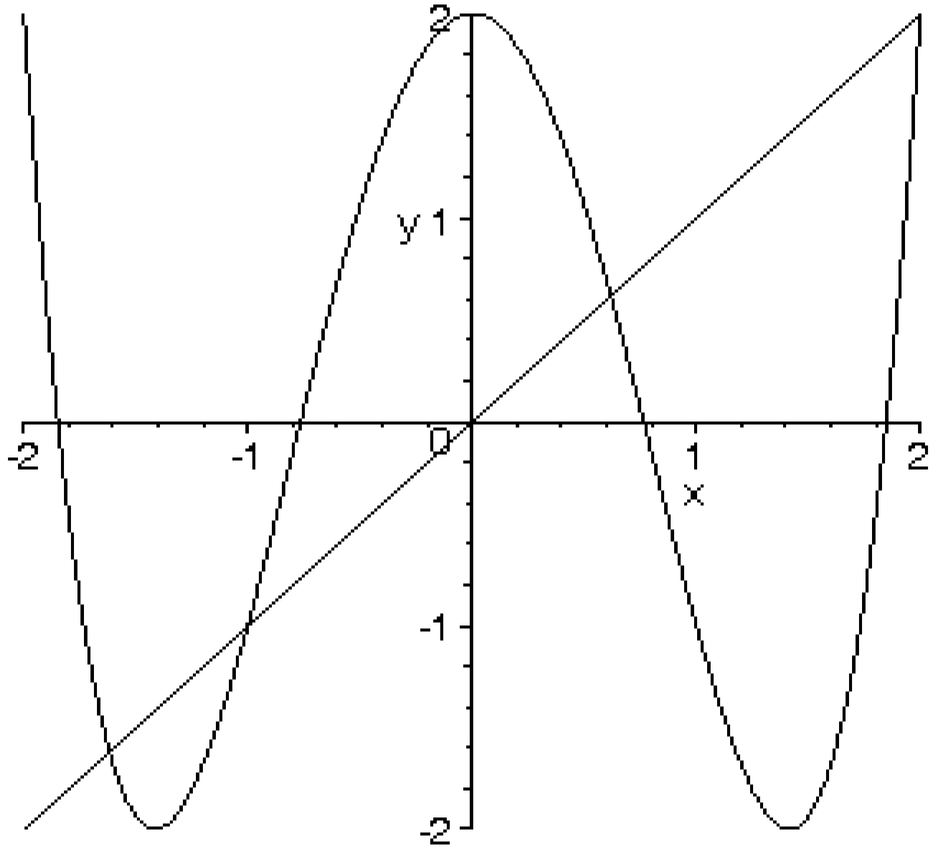


- We say that the interval [-2, 2] is *invariant* under Q_{-2}

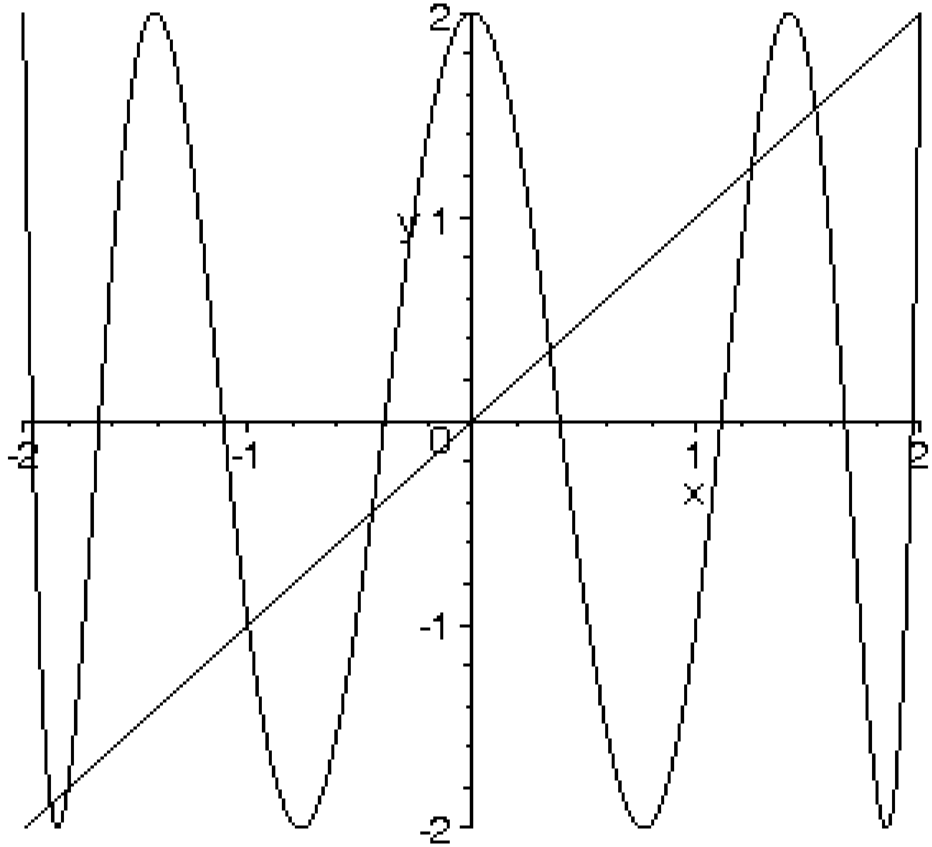
Stretch and fold

- Q_{-2} folds [-2, 2] over itself so that each point (except -2) is covered twice
- Q_{-2}^2 does to Q_{-2} what Q_{-2} does to x
- And so on …
- Repetitive stretching and folding of [-2, 2] onto itself gives rise to many periodic points

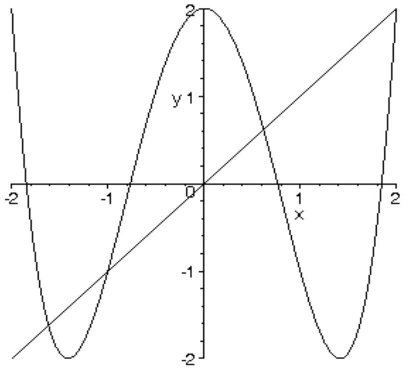
Q_{-2}^{2}



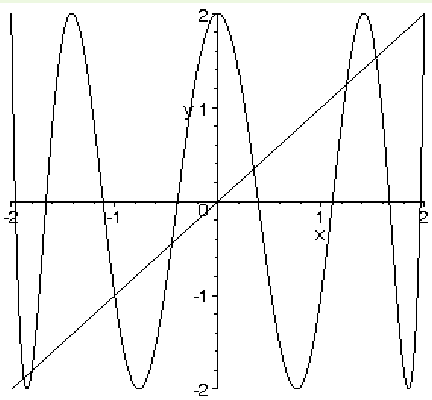
$Q_{-2}{}^{3}$



Q_{-2}^{2}



Q_{-2}^{3}



Observations

- Q_{-2}^{n} has _____ valleys, each taking [-2, 2] onto itself via a stretch - fold mapping
- What type of periodic and fixed points does Q_{-2}^{n} have?
- Since iterates never leave [-2, 2], we can consider the action of Q_{-2}^{p} on [-2, 2] for all p
- Infinitely many periodic orbits - yet still bounded by [-2, 2]

Observations (cont.)

- This is wild! There are infinitely many points that cycle (all in [-2, 2]), and yet there are infinitely many points that are repelled from these cycles (all in [-2, 2])
- The last set (the ones that aren't cycles) is bigger since there can only be a countable number of cycles, and yet there is an uncountable number of points in [-2, 2]

Countably infinite

- A set of numbers, S, is **countably infinite** if there is a 1-1 function which takes numbers in S onto each element of **N** = { 1, 2, 3, … }
- We then say that S has a "one-to-one correspondence" with the set of natural numbers

Example 1

- S = { 1, 4, 9, 16, 25, … } is countably infinite since $F(x) = x^2$ takes each point in **N** onto a number in S
- We can take $F{:}S \rightarrow \mathbf{N}$ or $F{:}\mathbf{N} \rightarrow S$ since F is necessarily invertible
- The "onto" is required so that we don't miscount (overlook a number)

Example 2

- There are the same number of integers as natural numbers (**Z** = { …, -3, -2, -1, 0, 1, 2, 3, … }
- Let F(x) =
  - -x/2 if x is even
  - (x-1)/2 if x is odd
- Then F:**N**→**Z** is 1-1

Countable and uncountable

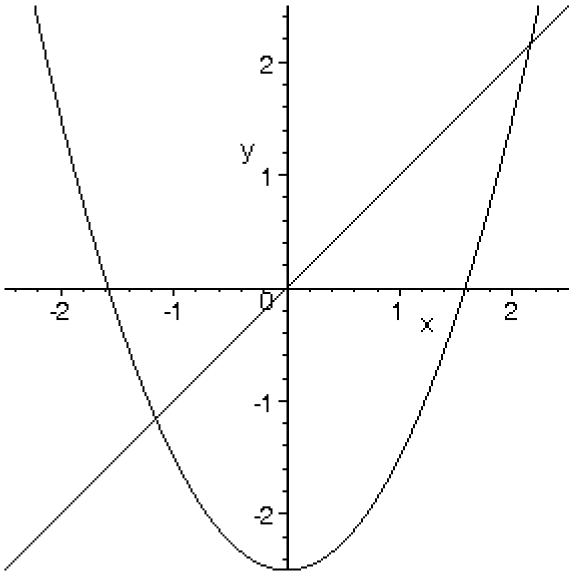
- A set is **countable** if it's finite or countably infinite
- A set is **uncountable** if it's not countable
- The set of rational numbers, **Q**, is a countable set
- The set of irrational numbers is uncountable

Back to $Q_{-2}(x)$

- Countably infinite points in $Q_{-2}(x)$ are periodic
- Most points in $Q_{-2}(x)$ are not (they remain in the interval, but where they go is a mystery!)

The case $c < -2$

- $\exists$ an interval $A_1 \ni Q_c(x) \notin [-2, 2]\ \forall\ x \in A_1$ (and $Q_c^n(x) \notin [-2, 2]\ \forall$ n)



Observations

- $\exists$ a set of numbers $A_2 \ni Q_c(A_2) = A_1$
- $A_2 \in [-2, 2]$ and $A_1 \cap A_2 = \varnothing$
- A_2 is 2 disjoint sub-intervals of [-2, 2]

Observations (cont.)

- With the n^{th} iterate, there are 2^n pieces squished onto [-2, 2] (and stuff diverging out of [-2, 2])
- For each iterate Q_c^n, $\exists$ a set with 2^n disjoint sub-intervals on [-2, 2] (call the set A_n) $\ni Q_c^n(A_n) \not\subset [-2, 2]$ and $A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n = \varnothing$ (the sets are disjoint!)

Non-diverging points

- A lot of points in [-2, 2] diverge
- How many, if any, don't?
- Denote the set that does not leave [-2, 2] under an arbitrary number of iterates by Λ
- Λ is a ________ set
- Just as $Q_{-2}(x)$ has a countable infinity of periodic repelling orbits, $Q_c(x)$ must as well when $c < -2$

Laptops

- Please bring laptops to class for the next 2 classes for an investigation into controlling chaos