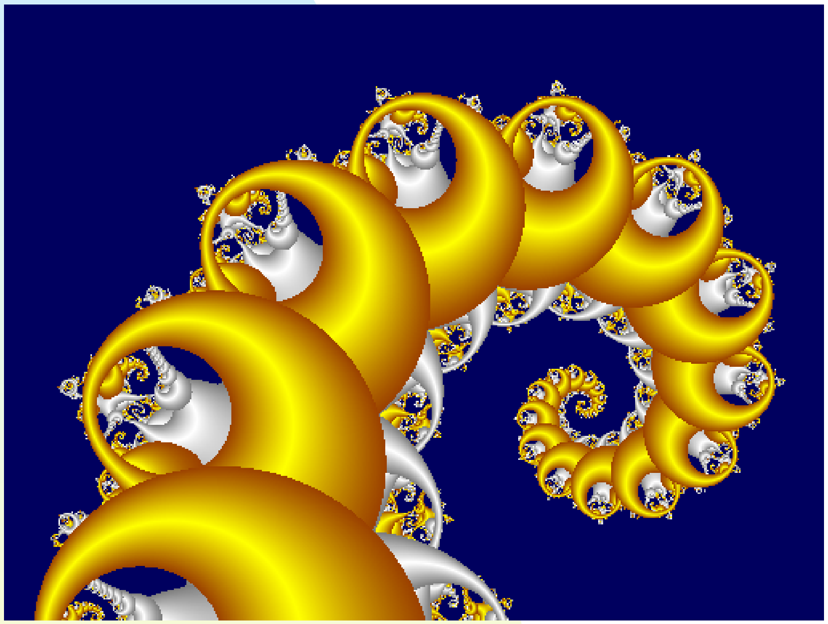
Session overview



- Fractional Brownian motion
- Announcements:
  - Project 3 due now
  - Project 4 assigned at the end of class, due Friday, 11:59 pm.

Review

- Brownian motion
  - Direct displacements
  - Simulated using Gaussian RV's
- Random midpoint displacement method
- Programs to calculate each

Hurst exponent

- Recall the equation for proper rescaling of Brownian motion: $X(rt) = (\sqrt{r})X(t)$
- Observe the power of r is ½
- This exponent is usually denoted *H* and sometimes called the *Hurst exponent*, after the hydrologist Hurst, who did some early work with Mandelbrot on scaling properties of river fluctuations

Fractional Brownian motion

- When we let *H* fall in the range $0 < H < 1$ we get what is called *fractional Brownian motion*, or fBm
  - $X(rt) = r^H X(t)$

- This is also recognized by letting $\mathrm{Var}\,(X(t_2) - X(t_1)) \propto \Delta t^{2H}$

Dimension of fBm

- Let r = 2: $X(t) = X(2t)/(2^H)$
- Suppose the graph of X(t) for $0 \le t \le 1$ is covered by N boxes of size *r*
- Now consider boxes of half the size, *r*/2
- For the interval $0 \le t \le ½$ the range of X(t) is $1/2^H$ times that of X(t) over the whole interval
- Will need $2N/2^H = 2^{1-H}N$ boxes of the smaller size to cover this half interval
- The same holds true for the other half interval

Dimension of fBm (cont.)

- The total needed is therefore $2 \cdot 2^{1-H}N = 2^{2-H}N$ boxes
- In general, we need $(2^{2-H})^k N$ boxes, of size $r/2^k$
- Determine the box-counting dimension.

- Characterization:
  - The case for $H = ½$ is ordinary Brownian motion (independent increments)
  - For $H > ½$ there is a positive correlation between increments
  - For $H < ½$ there is a negative correlation between increments
  - More natural looking landscapes will have an H value around 0.8, making D around 1.2

Generating fBm

- Use the random midpoint displacement method
- Start by choosing $X(0) = 0$ and $X(1)$ as a sample of a Gaussian random variable with mean 0 and variance σ^2
- Now compute $X(½)$ by averaging $X(0)$ and $X(1)$ and adding a random offset D_1 (Gaussian, with variance $2^{-2H}(1-2^{2H-2})\sigma^2$)

Why this variance for D_1?

- $Var(X(1)-X(0)) = \sigma^2$
- $X(½) = ½ (X(0) + X(1)) + D_1$
- $X(½) - X(0) = ½ (X(0) + X(1)) - X(0) + D_1$
  $= ½ (X(1) - X(0)) + D_1$
- $Var(X(½) - X(0))$ must be $(½)^{2H}\sigma^2$
- Since we are using independent random variables, $Var(½ (X(1) - X(0)) + D_1) = Var(½ (X(1) - X(0))) + Var(D_1)$
- $Var(½ (X(1) - X(0))) = ¼\sigma^2$
- This means $Var(D_1)$ must be $(½)^{2H}\sigma^2 - ¼\sigma^2$, or $2^{-2H}(1-2^{2H-2})\sigma^2$

Continuing with the process

- Value for $X(\frac{1}{4})$ is computed by averaging $X(0)$ and $X(\frac{1}{2})$ and adding a random offset D_2 (Gaussian, with variance $(2^{-2H})^2(1-2^{2H-2})\sigma^2$)
- $X(\frac{3}{4})$ is found by averaging $X(\frac{1}{2})$ and $X(1)$ and adding a similar random offset
- Continue in this manner; each further subdivision multiplies the variance of the offset by 2^{-2H}

Problem with this method

- For $H \neq \frac{1}{2}$ we don't get true fBm since the increments are not stationary
- $\mathrm{Var}\,(X(\frac{1}{2})-X(0)) = \mathrm{Var}\,(X(1) - X(\frac{1}{2})) = (\frac{1}{2})^{2H}\sigma^2$
- However, $\mathrm{Var}\,(X(\frac{3}{4}) - X(\frac{1}{4})) \neq (\frac{1}{2})^{2H}\sigma^2$, which it should if it were stationary

What is Var $(X(3/4)-X(1/4))$?

- $X(3/4)-X(1/4) = 1/2(X(1/2)+X(1)) + D_{21} - [1/2(X(0)+X(1/2)) + D_{22}] =$

  $1/2(X(1)-X(0)) + D_{21} - D_{22}$
- $Var(X(3/4)-X(1/4)) = 1/4 Var(X(1)-X(0)) + Var(D_{21}) + Var(D_{22}) =$

  $1/4\sigma^2 + 2(2^{-2H})^2(1-2^{2H-2})\sigma^2$

Example program

- `midpointfBm.cpp` has source code that implements the random midpoint displacement method for generating fractional Brownian motion
- However, the random midpoint displacement method must be tweaked for fractional Brownian motion. This is the goal of Project 4.

Project #4

- Implement one-dimensional fractional Brownian motion with successive random additions and lacunarity