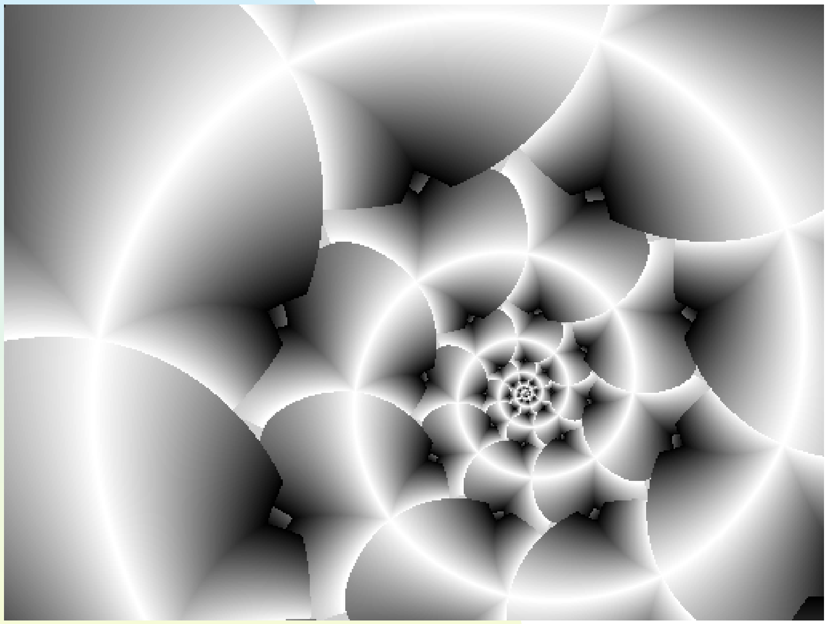
Session overview



- Simulating Brownian motion using Gaussian random variables
- Random midpoint displacement method
- Please pass in yesterday's quiz now.

Gaussian distribution

- Distribution of the total displacement is Gaussian
- The central limit theorem states that the sum or average of independent, identically distributed random events has a Gaussian distribution
- Use a Gaussian r.v. to represent the effect of an unknown number of displacements occuring over time

Proper rescaling

- Let $X(t)$ be a Gaussian random variable such that $X(0) = 0$ and $E((X(t_n))^2) = n$
- What if we double the time of the process? What can we say about $X(2t)$?
- $E(X^2(2t_n)) = 2n = 2\,E(X^2(t_n)) = E(2X^2(t_n))$
- $\therefore X^2(2t_n) = 2X^2(t_n)$ or $X(2t_n) = (\sqrt{2})X(t_n)$
- In general, $X(rt) = (\sqrt{r})X(t)$
- This is called a ***proper rescaling***

One way to simulate Brownian motion

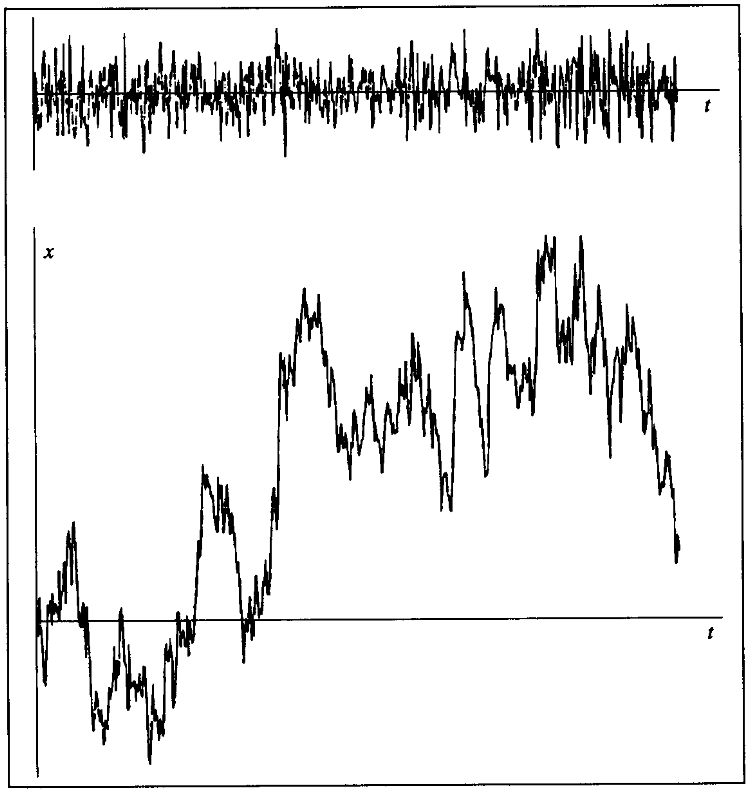
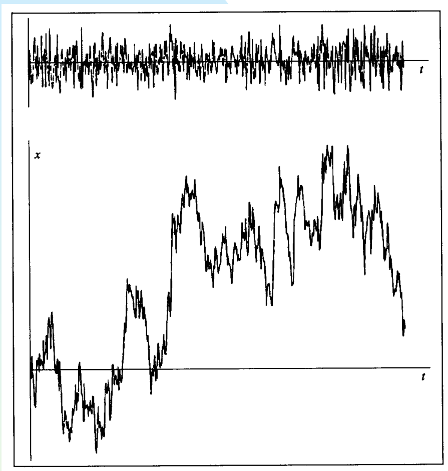


Figure 9.26 (p. 452 of PJS)

```
displacement = 0
moveTo (0, 0)
for i = 1 to N {
   displacement += Gaussian
         random number
   lineTo (i, displacement)
}
```

- See brownian.c

Dimension of Brownian motion



- Consider a part of the graph, with $\Delta t = 1$, $\Delta X = 1$
- Change Δt to $1/N$ and consider N subintervals
- $X(1/N) = 1/(\sqrt{N})X(1) = N^{-1/2}X(1)$
- $\therefore$ ΔX goes to $N^{-1/2}$
- The "area" covered by the trace is $\Delta X \Delta t = N^{-1/2}(1/N) = N^{-3/2} = 1/N^{3/2}$
- Comparing this to $1/N^D$ gives $D = 3/2 = 1.5$

Random midpoint displacement method

- The most popular method of producing Brownian motion
- Rather than generate locations sequentially in time, generate **the final location first**, and then recursively interpolate (with randomness) to fill in intermediate locations
- Extends to higher dimensions easily
  - For example, fractal interpolation can be used to generate altitudes of mountains in ranges

Random midpoint displacement method

- Start by choosing $X(0) = 0$ and $X(1)$, for some appropriate scale, as a sample of a Gaussian random variable with mean 0 and variance σ^2
  - This can be done by multiplying the standard Gaussian random number sample by σ, the standard deviation
- Now compute $X(\frac{1}{2})$ by averaging $X(0)$ and $X(1)$ and adding a random offset D_1 (Gaussian, with variance $\frac{1}{4}\sigma^2$)

Continuing with the process

- Value for X(¼) is computed by averaging X(0) and X(½) and adding a random offset D_2 (Gaussian, with variance 1/8 σ^2)
- X(¾) is found by averaging X(½) and X(1) and adding a similar random offset
- Continue in this manner; each further subdivision multiplies the variance of the offset by ½

Why a variance of $\frac{1}{4}\sigma^2$ for D_1?

- Recall that $E(X^2(t)) \propto \Delta t$
- $Var(X(1)-X(0)) = (1-0)\sigma^2$
- $X(\frac{1}{2}) = \frac{1}{2}(X(0) + X(1)) + D_1$
- $X(\frac{1}{2}) - X(0) = \frac{1}{2}(X(0) + X(1)) - X(0) + D_1 = \frac{1}{2}(X(1) - X(0)) + D_1$
- $Var(X(\frac{1}{2}) - X(0))$ must be $\frac{1}{2}\sigma^2$ (this comes from the Δt property)

Why a variance of $\frac{1}{4}\sigma^2$ for D_1? (cont.)

- Since we are using independent random variables, Var (½ (X(1) - X(0)) + D_1) = Var (½ (X(1) - X(0))) + Var (D_1)
- Var (½ (X(1) - X(0))) = $\frac{1}{4}\sigma^2$, since Var (aX) = a^2 Var (X)
- This means Var (D_1) must also be $\frac{1}{4}\sigma^2$, so multiply Gaussian random number by $\frac{1}{2}\sigma$

Example program

- **`midpointBrownian.c`** has source code that implements the random midpoint displacement method for generating Brownian motion

- Compare with **`brownian.c`**