



CSSE 230

How can we solve recurrence relations?
How many ways can we sort arrays?

$$T(N) = \begin{cases} \theta(N^{\log_b a}) & \text{if } a > b^k \\ \theta(N^k \log N) & \text{if } a = b^k \\ \theta(N^k) & \text{if } a < b^k \end{cases}$$

After today, you should be able to...
...write recurrences for code snippets
...solve recurrences using telescoping,
recurrence trees, and the master method

More on Recurrence Relations

A technique for analyzing recursive algorithms

Recap: Recurrence Relation

- ▶ An equation (or inequality) that relates the N^{th} element of a sequence to certain of its predecessors (recursive case)
- ▶ Includes an initial condition (base case)
- ▶ **Solution:** A function of N .

Example. Solve using backward substitution.

$$T(N) = 2T(N/2) + N$$

$$T(1) = 1$$

Solution strategies

Forward substitution Backward substitution

Simple
Sometimes can't solve
difficult relations

Recurrence trees

Visual
Great intuition for div-and-conquer

Telescoping

Widely applicable
Difficult to formulate
Not intuitive

Master Theorem

Immediate
Only for div-and-conquer
Only gives Big-Theta



Which
telescope?

Selection Sort: iterative version

```
static void selectionSort(int[] a) {  
    for (int last = a.length-1; last > 0; last--) {  
        int largest = a[0];  
        int largestPosition = 0;  
        for (int j=1; j<=last; j++) {  
            if (largest < a[j]) {  
                largest = a[j];  
                largestPosition = j;  
            }  
        }  
        a[largestPosition] = a[last];  
        a[last] = largest;  
    }  
}
```

What's N?

Selection Sort: recursive version

```
static void selectionSortRec(int[] a) {  
    selectionSortRec(a, a.length-1);  
}
```

```
static void selectionSortRec(int[] a, int last) {  
    if (last == 0) return;  
    int largest = a[0];  
    int largestPosition = 0;  
    for (int j=1; j<=last; j++) {  
        if (largest < a[j]) {  
            largest = a[j];  
            largestPosition = j;  
        }  
    }  
    a[largestPosition] = a[last];  
    a[last] = largest;  
    selectionSortRec(a, last-1);  
}
```

What's N?

Telescoping

- ▶ Basic idea: Set up equations so that when we sum all L sides and all R sides, we get an equation with lots of cancelation.
- ▶ Example: $T(1) = 0$, $T(N) = T(N - 1) + N - 1$

$$T(N) = \cancel{T(N-1)} + N - 1$$

$$\cancel{T(N-1)} = \cancel{T(N-2)} + N - 2$$

$$\cancel{T(N-2)} = \cancel{T(N-3)} + N - 3$$

$$\vdots$$

$$\cancel{T(2)} = \cancel{T(1)} + 1$$

$$\cancel{T(1)} = 0$$

$$T(N) = \sum_{i=1}^{N-1} i = \frac{(N-1)N}{2}$$



Telescoping

- ▶ In general, need to tweak the relation somehow so successive terms cancel
- ▶ Example: $T(1) = 1$, $T(N) = 2T(N/2) + N$
where $N = 2^k$ for some k
- ▶ Divide by N to get a “piece of the telescope”:

$$T(N) = 2T\left(\frac{N}{2}\right) + N$$

$$\implies \frac{T(N)}{N} = \frac{2T\left(\frac{N}{2}\right)}{N} + 1$$

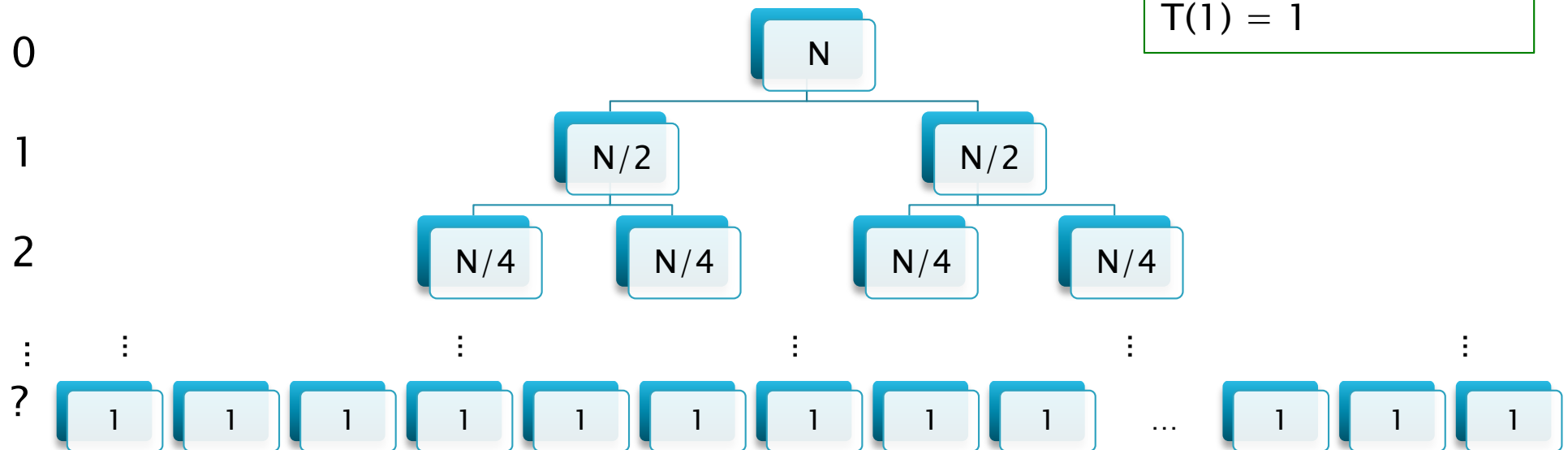
$$\implies \frac{T(N)}{N} = \frac{T\left(\frac{N}{2}\right)}{\frac{N}{2}} + 1$$

Etc.



Recursion tree

Level



Recurrence:

$$T(N) = 2T(N/2) + N$$

$$T(1) = 1$$

- How many nodes at level i ?
- How much work at level i ?
- Index of last level?

$$2^i$$

$$2^i (N/2^i) = N$$

$$\log_2 N$$

Total:
$$T(n) = \sum_{i=0}^{\log N} N = N(\log N + 1)$$

Master Theorem

- ▶ For Divide-and-conquer algorithms
 - Divide data into one or more parts **of the same size**
 - Solve problem on one or more of those parts
 - Combine "parts" solutions to solve whole problem
- ▶ Examples
 - Binary search
 - Merge Sort
 - MCSS recursive algorithm we studied last time

Theorem 7.5 in Weiss

Master Theorem

- ▶ For any recurrence in the form:

$$T(N) = aT(N/b) + \theta(N^k)$$

with $a \geq 1, b > 1$

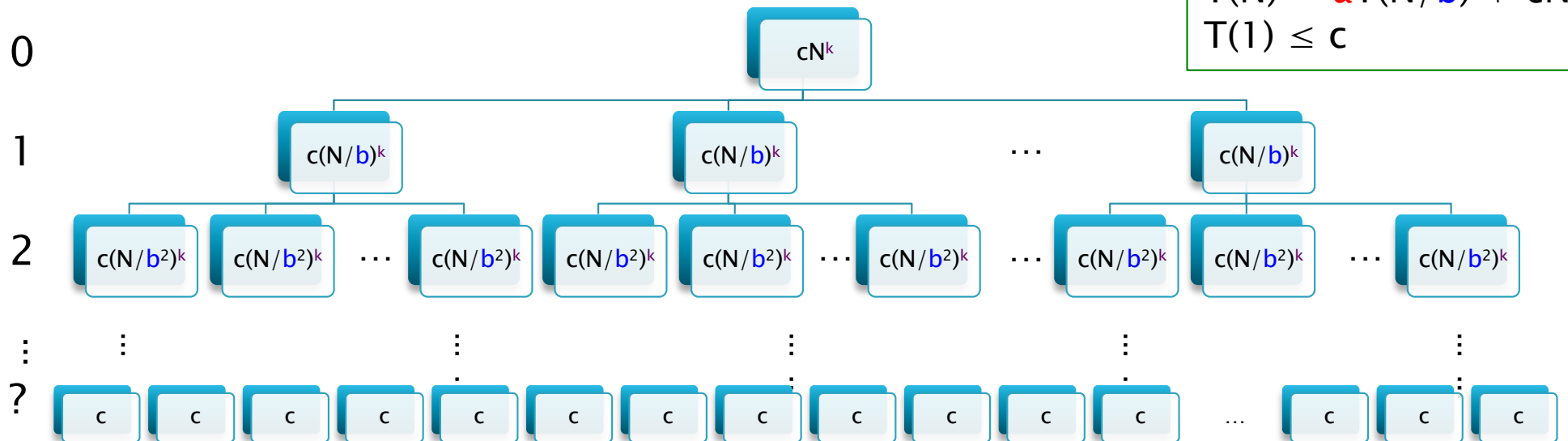
- ▶ The solution is

$$T(N) = \begin{cases} \theta(N^{\log_b a}) & \text{if } a > b^k \\ \theta(N^k \log N) & \text{if } a = b^k \\ \theta(N^k) & \text{if } a < b^k \end{cases}$$

Example: $2T(N/4) + N$

Master Recurrence Tree

Level



- How many nodes at level i ?
- How much work at level i ?
- Index of last level?

$$a^i$$

$$a^i c(N/b^i)^k = cN^k (a/b^k)^i$$

$$\log_b N$$

Summation:
$$T(N) \leq cN^k \sum_{i=0}^{\log_b N} \left(\frac{a}{b^k}\right)^i$$

Interpretation

- ▶ Upper bound on work at level i : $cN^k \left(\frac{a}{b^k}\right)^i$
- ▶ a = “Rate of subproblem proliferation” 😞
- ▶ b^k = “Rate of work shrinkage” 😊

Case	😞 $a < b^k$ 😊	😞 $a = b^k$ 😊	😞 $a > b^k$ 😊
As level i increases...	😊 work goes down!	😐 work stays same	😞 work goes up!
$T(N)$ dominated by work done at...	Root of tree	Every level similar	Leaves of tree
Master Theorem says $T(N)$ in...	$\Theta(N^k)$	$\Theta(N^k \log N)$	$\Theta(N^{\log_b a})$

Master Theorem – End of Proof

- ▶ Case 1. $a < b^k$

$$cN^k \left(\frac{1 - (a/b^k)^{\log_b N + 1}}{1 - (a/b^k)} \right) \approx cN^k \left(\frac{1}{1 - (a/b^k)} \right)$$

$$cN^k \sum_{i=0}^{\log_b N} \left(\frac{a}{b^k} \right)^i$$

- ▶ Case 2. $a = b^k$

$$cN^k \sum_{i=0}^{\log_b N} 1 = cN^k (\log_b N + 1)$$

- ▶ Case 3. $a > b^k$

$$cN^k \left(\frac{(a/b^k)^{\log_b N + 1} - 1}{(a/b^k) - 1} \right) \approx cN^k (a/b^k)^{\log_b N} = ca^{\log_b N} = cN^{\log_b a}$$

Summary: Recurrence Relations

- ▶ Analyze code to determine relation
 - Base case in code gives base case for relation
 - Number and “size” of recursive calls determine recursive part of recursive case
 - Non-recursive code determines rest of recursive case
- ▶ Apply a strategy
 - Guess and check (substitution)
 - Telescoping
 - Recurrence tree
 - Master theorem

Sorting overview

Quick look at several sorting methods

Focus on quicksort

Quicksort average case analysis

Elementary Sorting Methods

- ▶ Name as many as you can
- ▶ How does each work?
- ▶ Running time for each (sorting N items)?
 - best
 - worst
 - average
 - extra space requirements
- ▶ Spend 10 minutes with a group of 2-3, answering these questions. Then we will summarize

Put list on board

INEFFECTIVE SORTS

```
DEFINE HALFHEARTEDMERGESORT(LIST):  
  IF LENGTH(LIST) < 2:  
    RETURN LIST  
  PIVOT = INT(LENGTH(LIST) / 2)  
  A = HALFHEARTEDMERGESORT(LIST[:PIVOT])  
  B = HALFHEARTEDMERGESORT(LIST[PIVOT:])  
  // UMMMMM  
  RETURN[A, B] // HERE. SORRY.
```

```
DEFINE FASTBOGOSORT(LIST):  
  // AN OPTIMIZED BOGOSORT  
  // RUNS IN O(N LOG N)  
  FOR N FROM 1 TO LOG(LENGTH(LIST)):  
    SHUFFLE(LIST):  
    IF ISSORTED(LIST):  
      RETURN LIST  
  RETURN "KERNEL PAGE FAULT (ERROR CODE: 2)"
```

```
DEFINE JOBINTERVIEWQUICKSORT(LIST):  
  OK SO YOU CHOOSE A PIVOT  
  THEN DIVIDE THE LIST IN HALF  
  FOR EACH HALF:  
    CHECK TO SEE IF IT'S SORTED  
    NO, WAIT, IT DOESN'T MATTER  
    COMPARE EACH ELEMENT TO THE PIVOT  
    THE BIGGER ONES GO IN A NEW LIST  
    THE EQUAL ONES GO INTO, UH  
    THE SECOND LIST FROM BEFORE  
  HANG ON, LET ME NAME THE LISTS  
  THIS IS LIST A  
  THE NEW ONE IS LIST B  
  PUT THE BIG ONES INTO LIST B  
  NOW TAKE THE SECOND LIST  
  CALL IT LIST, UH, A2  
  WHICH ONE WAS THE PIVOT IN?  
  SCRATCH ALL THAT  
  IT JUST RECURSIVELY CALLS ITSELF  
  UNTIL BOTH LISTS ARE EMPTY  
  RIGHT?  
  NOT EMPTY, BUT YOU KNOW WHAT I MEAN  
  AM I ALLOWED TO USE THE STANDARD LIBRARIES?
```

```
DEFINE PANICSORT(LIST):  
  IF ISSORTED(LIST):  
    RETURN LIST  
  FOR N FROM 1 TO 10000:  
    PIVOT = RANDOM(0, LENGTH(LIST))  
    LIST = LIST[PIVOT:] + LIST[:PIVOT]  
    IF ISSORTED(LIST):  
      RETURN LIST  
  IF ISSORTED(LIST):  
    RETURN LIST  
  IF ISSORTED(LIST): // THIS CAN'T BE HAPPENING  
    RETURN LIST  
  IF ISSORTED(LIST): // COME ON COME ON  
    RETURN LIST  
  // OH JEEZ  
  // I'M GONNA BE IN SO MUCH TROUBLE  
  LIST = []  
  SYSTEM("SHUTDOWN -H +5")  
  SYSTEM("RM -RF ./")  
  SYSTEM("RM -RF ~/*")  
  SYSTEM("RM -RF /")  
  SYSTEM("RD /S /Q C:\*") // PORTABILITY  
  RETURN [1, 2, 3, 4, 5]
```

Stacksort connects to StackOverflow, searches for “sort a list”, and downloads and runs code snippets until the list is sorted.