CSSE 230

How can we solve recurrence relations?
How many ways can we sort arrays?

$$T(N) = \begin{cases} \theta(N^{log_b a}) & \text{if } a > b^k \\ \theta(N^k log N) & \text{if } a = b^k \\ \theta(N^k) & \text{if } a < b^k \end{cases}$$

After today, you should be able to…
…write recurrences for code snippets
…solve recurrences using telescoping, recurrence trees, and the master method

More on Recurrence Relations

A technique for analyzing recursive algorithms

Recap: Recurrence Relation

- An equation (or inequality) that relates the N^{th} element of a sequence to certain of its predecessors (recursive case)
- Includes an initial condition (base case)
- **Solution**: A function of N.

Example. Solve using backward substitution.

$$T(N) = 2T(N/2) + N$$
$$T(1) = 1$$

Solution strategies

Forward substitution
Backward substitution

Simple
Sometimes can't solve difficult relations

Recurrence trees

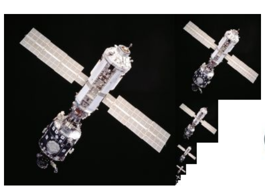
Visual
Great intuition for div-and-conquer

Telescoping

Widely applicable
Difficult to formulate
Not intuitive

Master Theorem

Immediate
Only for div-and-conquer
Only gives Big-Theta



Which telescope?

http://wiki.ogre3d.org/Mipmapping

Selection Sort: iterative version

```
static void selectionSort(int[] a) {
    for (int last = a.length-1; last > 0; last--) {
        int largest = a[0];
        int largestPosition = 0;
        for (int j=1; j<=last; j++) {
            if (largest < a[j]) {
                largest = a[j];
                largestPosition = j;
            }
        }
        a[largestPosition] = a[last];
        a[last] = largest;
    }
}
```

What's N?

Selection Sort: recursive version

```
static void selectionSortRec(int[] a) {
    selectionSortRec(a, a.length-1);
}

static void selectionSortRec(int[] a, int last) {
    if (last == 0) return;
    int largest = a[0];
    int largestPosition = 0;
    for (int j=1; j<=last; j++) {
        if (largest < a[j]) {
            largest = a[j];
            largestPosition = j;
        }
    }
    a[largestPosition] = a[last];
    a[last] = largest;
    selectionSortRec(a, last-1);
}
```

What's N?

Telescoping

- Basic idea: Set up equations so that when we sum all L sides and all R sides, we get an equation with lots of cancelation.
- Example: T(1) = 0, T(N) = T(N – 1) + N – 1

$$\begin{aligned} T(N) &= \cancel{T(N-1)} + N - 1 \\ \cancel{T(N-1)} &= \cancel{T(N-2)} + N - 2 \\ \cancel{T(N-2)} &= \cancel{T(N-3)} + N - 3 \\ &\vdots \\ \cancel{T(2)} &= \cancel{T(1)} + 1 \\ \cancel{T(1)} &= 0 \end{aligned}$$

$$T(N) = \sum_{i=1}^{N-1} i \qquad = \frac{(N-1)N}{2}$$

Telescoping

- In general, need to tweak the relation somehow so successive terms cancel

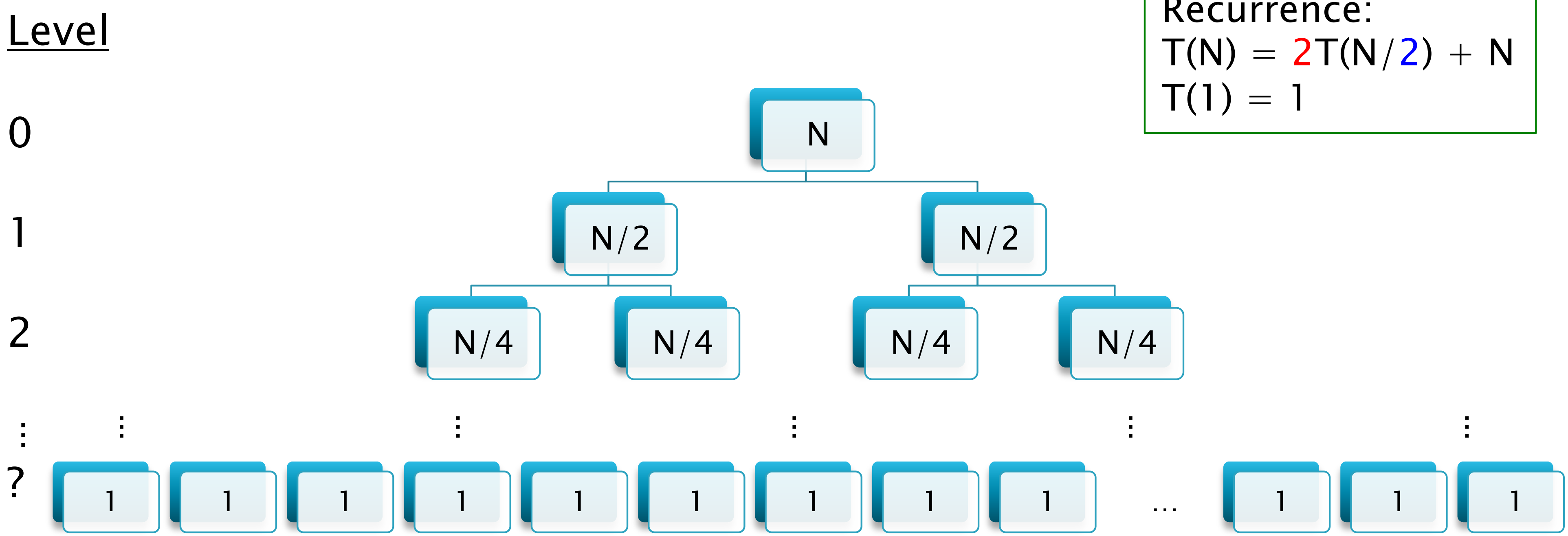
- Example: T(1) = 1, T(N) = 2T(N/2) + N
  where N = 2^k for some k
- Divide by N to get a "piece of the telescope":

$$
\begin{aligned}
T(N) &= 2T(\frac{N}{2}) + N \\
\Longrightarrow \frac{T(N)}{N} &= \frac{2T(\frac{N}{2})}{N} + 1 \\
\Longrightarrow \frac{T(N)}{N} &= \frac{T(\frac{N}{2})}{\frac{N}{2}} + 1
\end{aligned}
$$

Etc.

Recursion tree

Recurrence:
$T(N) = 2T(N/2) + N$
$T(1) = 1$



- How many nodes at level i? 2^i
- How much work at level i? $2^i (N/2^i) = N$
- Index of last level? $\log_2 N$

Total: $$T(n) = \sum_{i=0}^{\log N} N = N(\log N + 1)$$

Master Theorem

- For Divide-and-conquer algorithms
  - Divide data into one or more parts of the same size
  - Solve problem on one or more of those parts
  - Combine "parts" solutions to solve whole problem
- Examples
  - Binary search
  - Merge Sort
  - MCSS recursive algorithm we studied last time

Theorem 7.5 in Weiss

Master Theorem

- For any recurrence in the form:

$$T(N) = aT(N/b) + \theta(N^k)$$

$$\text{with } a \geq 1, b > 1$$

- The solution is

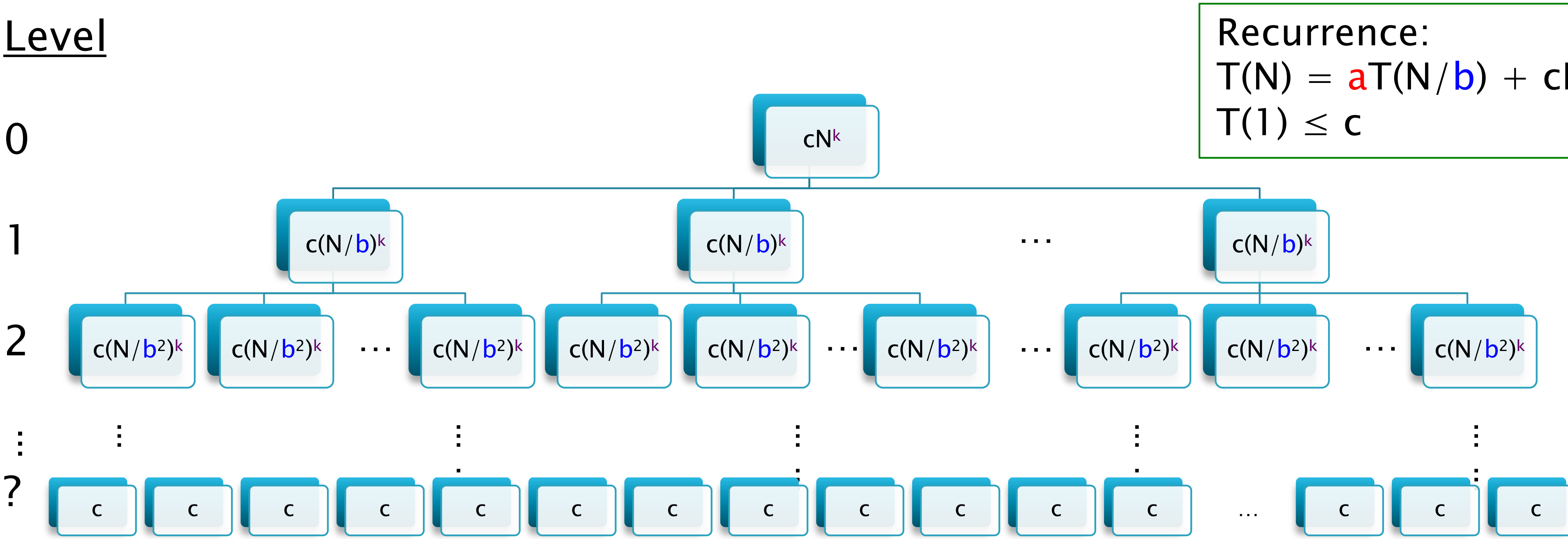
$$T(N) = \begin{cases} \theta(N^{log_b a}) & \text{if } a > b^k \\ \theta(N^k logN) & \text{if } a = b^k \\ \theta(N^k) & \text{if } a < b^k \end{cases}$$

Example: 2T(N/4) + N

Theorem 7.5 in Weiss

Master Recurrence Tree

Recurrence:
$T(N) = aT(N/b) + cN^k$
$T(1) \leq c$



- How many nodes at level i? a^i
- How much work at level i? $a^i c(N/b^i)^k = cN^k(a/b^k)^i$
- Index of last level? $\log_b N$

Summation: $$T(N) \leq cN^k \sum_{i=0}^{\log_b N} \left(\frac{a}{b^k}\right)^i$$

Interpretation

- Upper bound on work at level i: $cN^k \left(\frac{a}{b^k}\right)^i$

- a = "Rate of subproblem proliferation" 😟
- b^k = "Rate of work shrinkage" 😃

Case	😟 $a < b^k$ 😃	😟 $a = b^k$ 😃	😟 $a > b^k$ 😃
As level i increases…	😃 work goes down!	😐 work stays same	😟 work goes up!
T(N) dominated by work done at…	Root of tree	Every level similar	Leaves of tree
Master Theorem says T(N) in…	$\Theta(N^k)$	$\Theta(N^k \log N)$	$\Theta(N^{\log_b a})$

Master Theorem – End of Proof

$$cN^k \sum_{i=0}^{\log_b N} \left(\frac{a}{b^k}\right)^i$$

- Case 1. a < b^k

$$cN^k \left(\frac{1-(a/b^k)^{\log_b N+1}}{1-(a/b^k)}\right) \approx cN^k \left(\frac{1}{1-(a/b^k)}\right)$$

- Case 2. a = b^k

$$cN^k \sum_{i=0}^{\log_b N} 1 = cN^k(\log_b N + 1)$$

- Case 3. a > b^k

$$cN^k \left(\frac{(a/b^k)^{\log_b N+1} - 1}{(a/b^k) - 1}\right) \approx cN^k (a/b^k)^{\log_b N} = ca^{\log_b N} = cN^{\log_b a}$$

Summary: Recurrence Relations

- Analyze code to determine relation
  - Base case in code gives base case for relation
  - Number and "size" of recursive calls determine recursive part of recursive case
  - Non-recursive code determines rest of recursive case
- Apply a strategy
  - Guess and check (substitution)
  - Telescoping
  - Recurrence tree
  - Master theorem

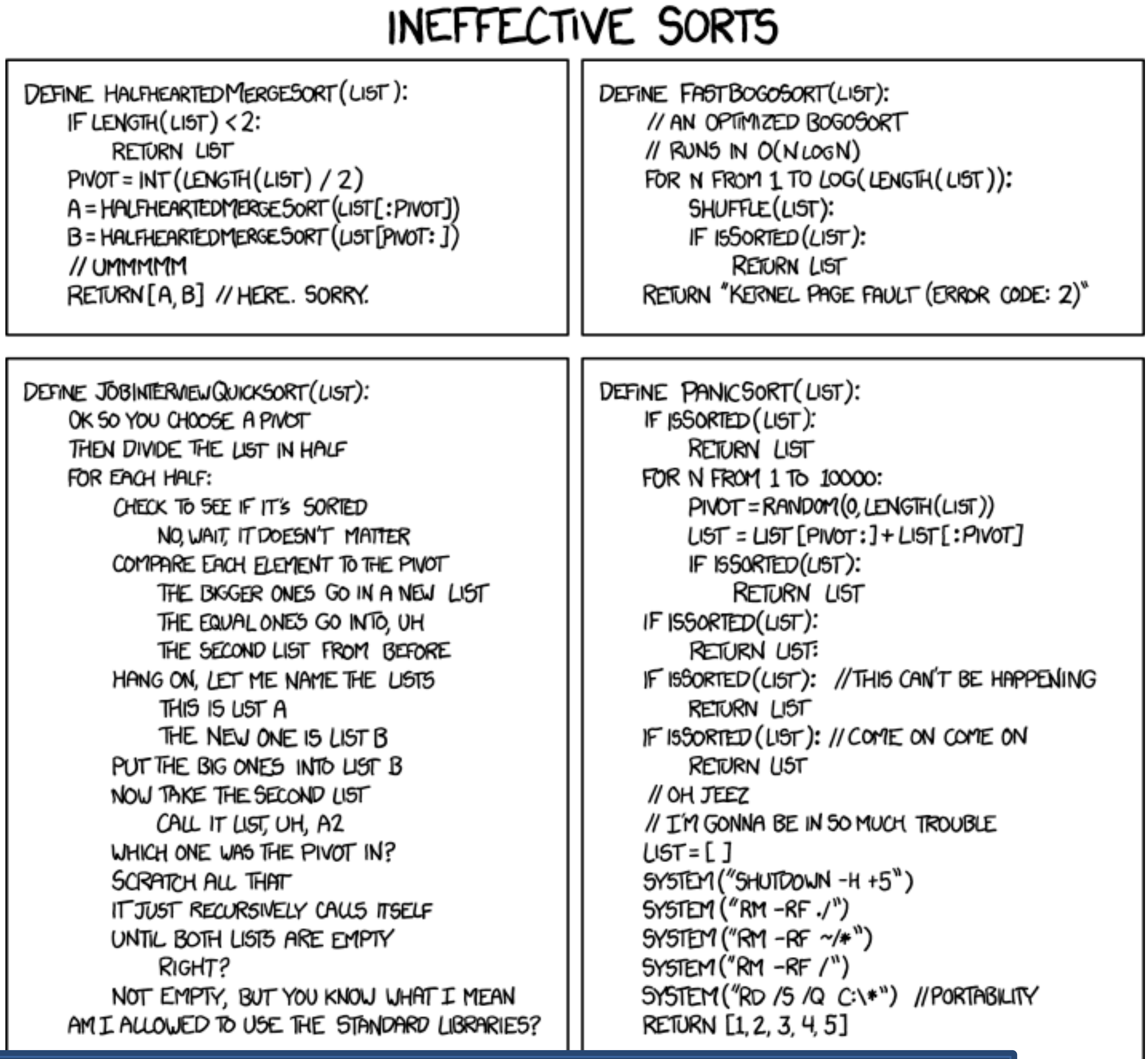
Sorting overview

- Quick look at several sorting methods
- Focus on quicksort
- Quicksort average case analysis

Elementary Sorting Methods

- Name as many as you can
- How does each work?
- Running time for each (sorting N items)?
  - best
  - worst
  - average
  - extra space requirements
- Spend 10 minutes with a group of 2-3, answering these questions. Then we will summarize

Put list on board



http://www.xkcd.com/1185/

Stacksort connects to StackOverflow, searches for "sort a list", and downloads and runs code snippets until the list is sorted.