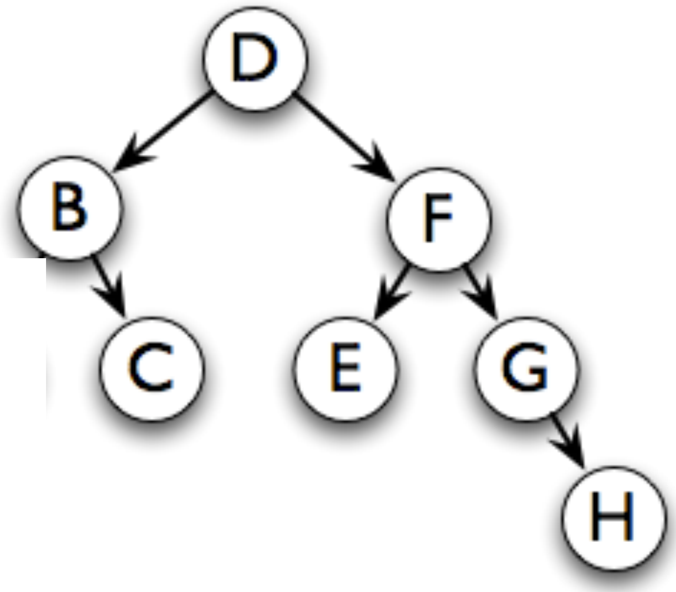


CSSE 230 Day 12

Height-Balanced Trees

After today, you should be able to…
…give the minimum number of nodes in a height-balanced tree
…explain why the height of a height-balanced trees is O(log n)
…help write an induction proof

Today's Agenda

- Announcements
  - ~~EditorTrees team preferences survey due 5 PM~~
  - HW 4 due tonight
  - ~~Also Doublets partner evaluation survey~~
  - Exam 2 (programming only) in class on Wed
    ~~You'll have about 85 minutes for the exam~~

- Another induction example
- Recap: The need for balanced trees
- Analysis of worst case for height-balanced (AVL) trees

A useful result… by way of induction

Q3

- Recall the definition of the Fibonacci numbers:
  - $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$
- List F_0 through F_{10} now.
- Prove the closed form:

7.8 Prove by induction the formula

$$F_N = \frac{1}{\sqrt{5}}\left(\left(\frac{(1+\sqrt{5})}{2}\right)^N - \left(\frac{1-\sqrt{5}}{2}\right)^N\right)$$

Recall: How to show that property P(n) is true for all $n \geq n_0$:

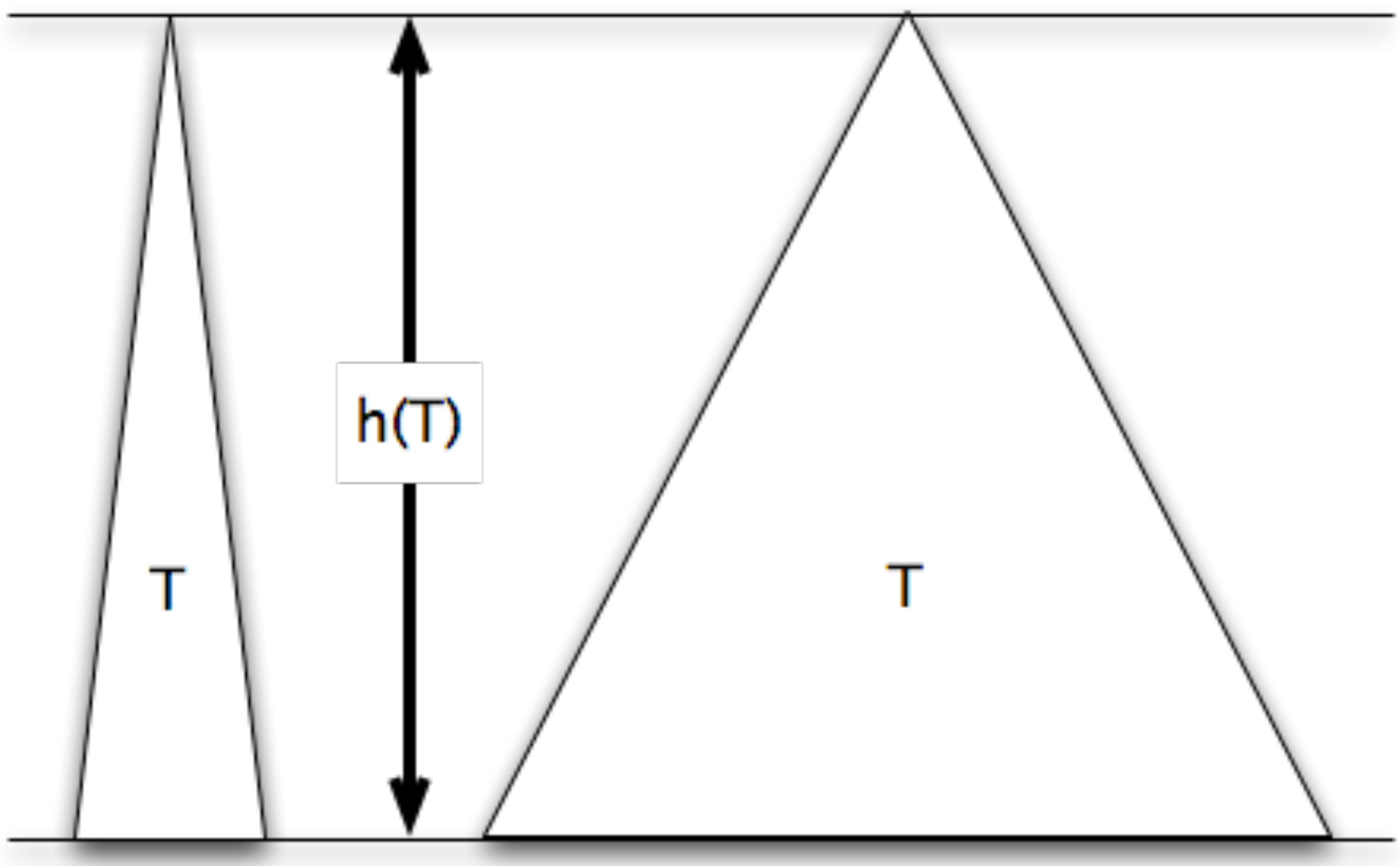
(1) Show the base case(s) directly

(2) Show that if P(j) is true for all j with $n_0 \leq j < k$, then P(k) is true also

Details of step 2:

a. Fix "arbitrary but specific" k ≥ ________.
b. Write the induction hypothesis: assume P(j) is true $\forall j : n_0 \leq j < k$
c. Prove P(k), using the induction hypothesis.

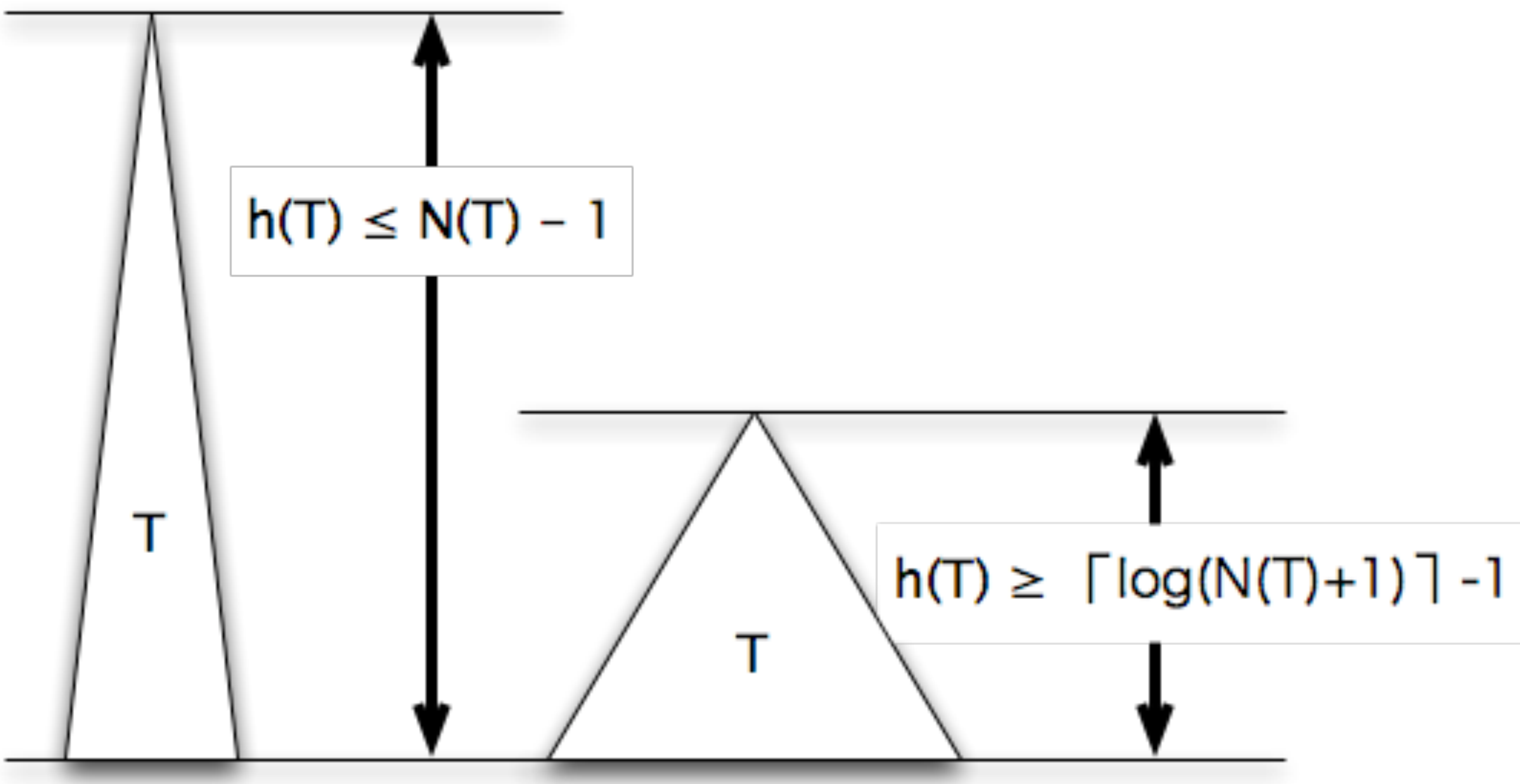
Review: The number of nodes in a tree with height h(T) is bounded



$N(T) \geq h(T) + 1$

$N(T) \leq 2^{h(T)+1} - 1$

Review: Therefore the height of a tree with N(T) nodes is also bounded



We want to keep trees balanced so that the run time of BST algorithms is minimized

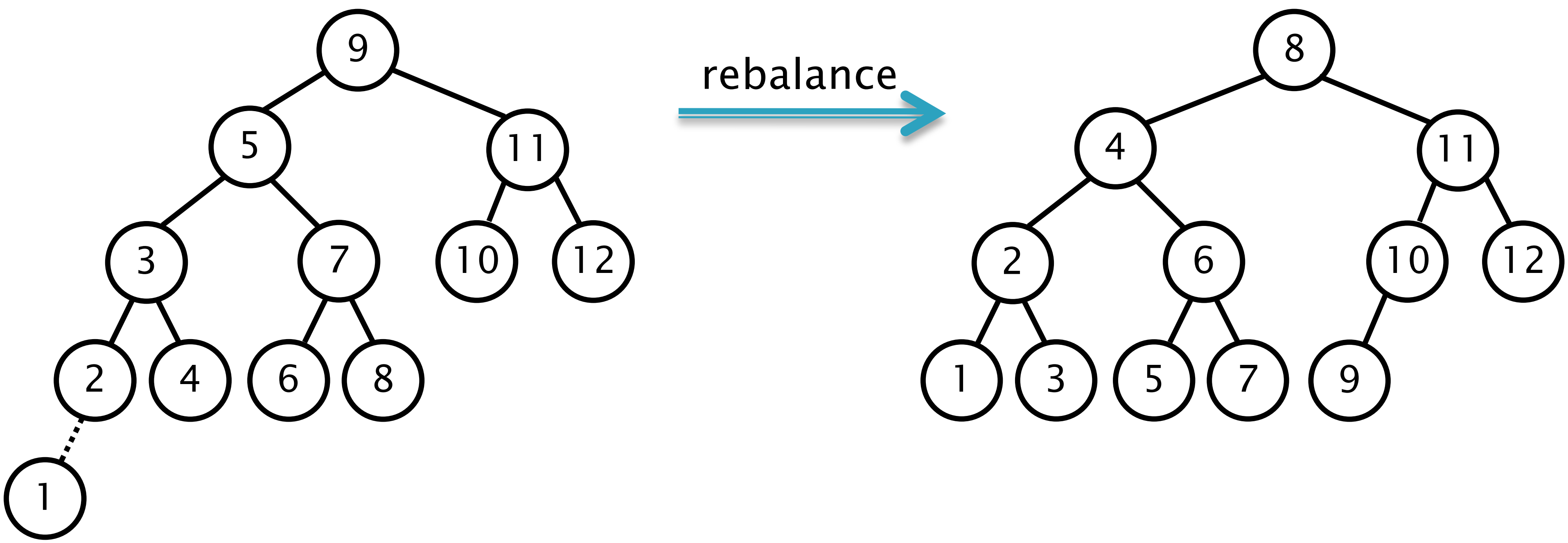
Q4

- BST algorithms are O(h(T))
- Minimum value of h(T) is $\lceil \log(N(T) + 1) \rceil - 1$
- Should we rearrange the tree after an insertion to guarantee that h(T) is always **minimized**?
  - Maintain "Complete balance"

But keeping complete balance is too expensive!

Q5

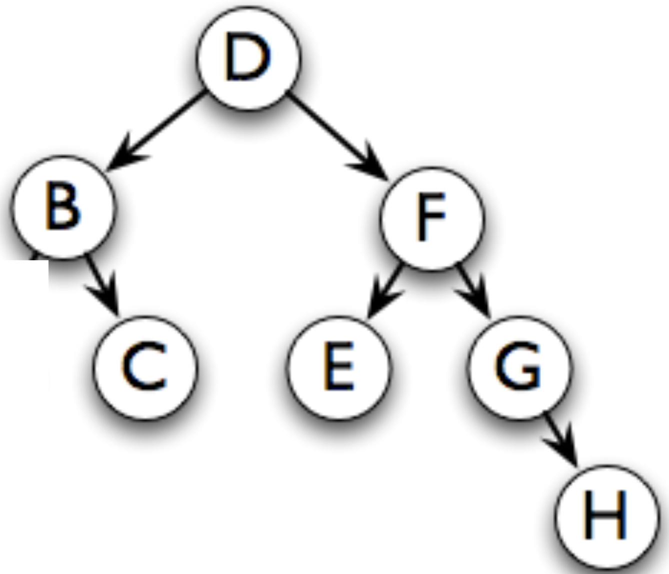
- Consider inserting 1 in the following tree.
- What does it take to get back to complete balance?
- Keeping completely balanced is too expensive:
  - O(N) to rebalance after insertion or deletion



Solution: Height Balanced Trees (less is more)

Height-Balanced Trees have subtrees whose heights differ by at most 1

Q6



Still height-balanced?

More precisely, a binary tree T is height balanced if

T is empty, or if

| **height**(T_L) – **height**(T_R) | ≤ 1, and

T_L and T_R are both height balanced.

What is the tallest (worst) height-balanced tree with N nodes?

Q7

Is it taller than a completely balanced tree?

- Consider the dual concept: find the minimum number of nodes for height h.

A binary search tree **T** is height balanced if

T is empty, or if

| **height(** T_L **)** – **height(** T_R **)** | ≤ **1**, and

T_L and T_R are both height balanced.

An AVL tree is a height-balanced BST that maintains balance using "rotations"

Q8-9

- Named for authors of original paper, **A**delson-**V**elskii and **L**andis (1962). (Russian)
- Max. height of an AVL tree with **N** nodes is: **H < 1.44 log (N+2) – 1.328 = O(log N)**

Our goal is to rebalance an AVL tree after insert/delete in O(log n) time

- Why?
- Worst cases for BST operations are **O(h(T))**
  - find, insert, and delete
- **h**(T) can vary from **O(log N)** to **O(N)**

- Height of a height-balanced tree is **O(log N)**

- So if we can rebalance after insert or delete in **O(log N)** time, then **all** operations are **O(log N)**