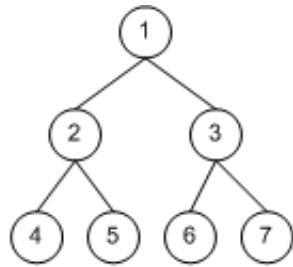


(a)



(b)

CSSE 230 Day 11

Size vs height in a Binary Tree

After today, you should be able to...

... use the relationship between the size and height of a tree to find the maximum and minimum number of nodes a binary tree can have

...understand the idea of mathematical induction as a proof technique

Some meme humor

If pants wore pants...
would they wear them



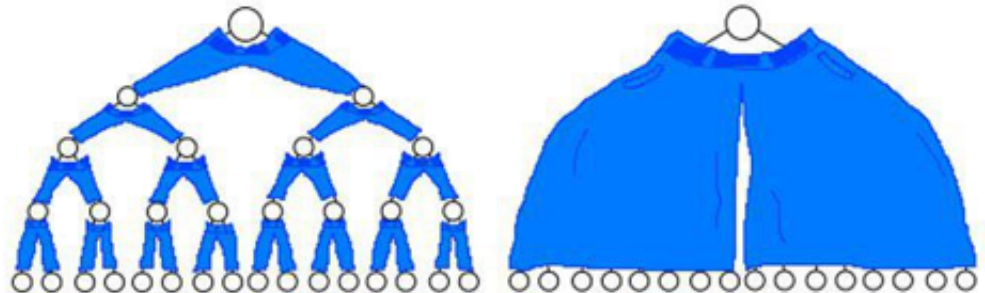
like this? or like this?

If a binary tree wore pants, would it wear them

like this

or

like this?



Other announcements

- Today:
 - Size vs height of trees: patterns and proofs
- Wrapping up the BST assignment, and worktime.

Extreme Trees

- A tree with the maximum number of nodes for its height is a **full tree**.
- full binary tree – each non-leaf node has exactly two children, all leaves at same level.
- A tree with the minimum number of nodes for its height is called **degenerate**
- Height matters!
 - Recall that the algorithms for search, insertion, and deletion in a binary search tree are $O(h(T))$

Proving a Universal Statement

- Example:

Open statement $S(n)$

$$\text{For all integers } n \geq 0, \sum_{i=1}^n i = \frac{n(n+1)}{2}.$$

- “for all” (\forall) is called the universal qualifier
- How to prove?
 - Can't do it one-by-one: there are infinitely many statements to prove!
 - Could try direct proof: show $\forall n S(n)$
 - Typically, pick *arbitrary but specific* n and prove using logic
 - But, often easier to use **induction**: show $S(0)$ and $\forall k (S(k) \rightarrow S(k + 1))$

Mathematical Induction

To prove that $P(n)$ is true for all $n \geq n_0$:

1. **Basis step**: Prove that $P(n_0)$ is true (base case), and
2. **Induction step**: Prove that if $P(k)$ is true for any $k \geq n_0$, then $P(k+1)$ is also true.

[This part of the proof must work for all such k !]

$$\left(P(n_0) \ \& \ \forall k (P(k) \rightarrow P(k+1)) \right) \rightarrow \forall n P(n)$$

- Note: we still need to prove a universal statement! But the advantage is that we're allowed to assume the **induction hypothesis** (truth of the "previous case" $P(k)$) in proving the "next case" $P(k+1)$.
- Example: prove the arithmetic sum formula

To prove recursive properties (on trees), we use a technique called mathematical induction

- Actually, we use a variant called *strong induction* :



The former
governor of
California

Strong Induction

$$\left(P(n_0) \ \& \ \forall k \left((P(n_0) \ \& \ \dots \ \& \ P(k)) \rightarrow P(k + 1) \right) \right) \rightarrow \forall n \ P(n)$$

- Strengthen the induction hypothesis
- Rather than assume truth of just the previous case $P(k)$, assume truth of **all** previous cases $P(n_0), P(n_0 + 1), P(n_0 + 2), \dots, P(k)$.

Strong Induction

- To prove that $p(n)$ is true for all $n \geq n_0$:
 - Prove that $p(n_0)$ is true (base case), and
 - For all $k > n_0$, prove that if we assume $p(j)$ is true for $n_0 \leq j < k$, then $p(k)$ is also true
- An analogy:
 - Regular induction uses the previous domino to knock down the next
 - Strong induction uses all the previous dominos to knock down the next!
- Example: prove the upper bound on $N(T)$ in terms of $h(T)$