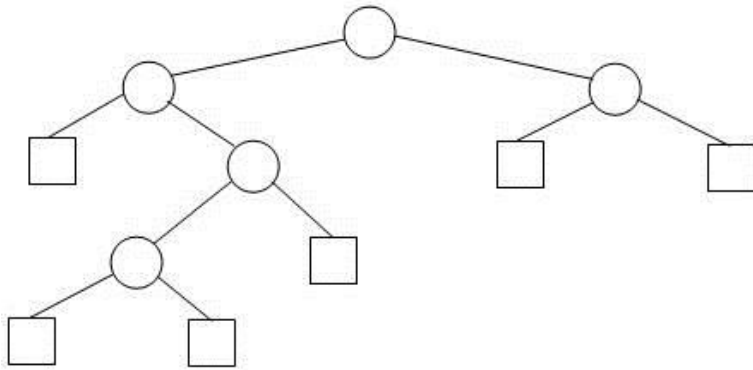# CSSE 230

Extended Binary Trees
Recurrence relations

After today, you should be able to…
…explain what an extended binary tree is
…solve simple recurrences using guess-and-check
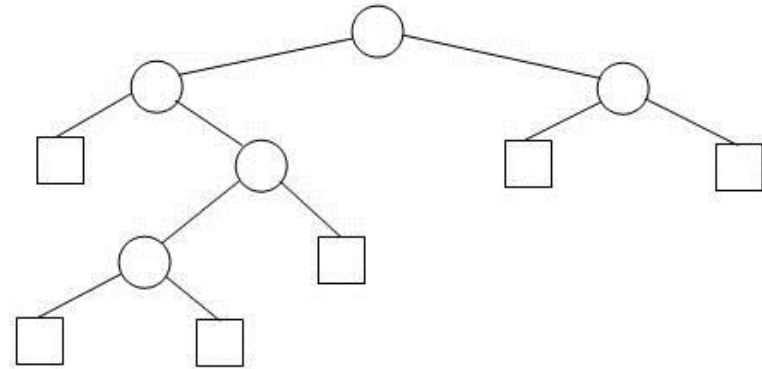
Reminders/Announcements

▸ Today:
- Extended Binary Trees (basis for much of WA8, which includes induction proofs and no programming)

- Recurrence relations, part 1

- EditorTrees worktime?

Extended Binary Trees (EBT's)

Bringing new life to Null nodes!

An Extended Binary Tree (EBT) just has *null* external nodes as leaves

▸ Not a single NULL_NODE,
  but many NULL_NODEs

▸ An Extended Binary tree is either
  ◦ an *external (null) node*, or
  ◦ an (**internal**) root node and two
    EBTs T_L and T_R.

▸ We draw internal nodes as circles and external nodes as squares.
  ◦ Generic picture and detailed picture.

▸ This is simply an alternative way of viewing binary trees, in which we view the external nodes as "places" where a search can end or an element can be inserted.

A property of EBTs

▸ **Property** P(N): For any N>=0, any EBT with N internal nodes has _____ external nodes.

▸ **Prove by strong induction**, based on the recursive definition.
  ◦ A notation for this problem: IN(T), EN(T)

Hint (reminder): Find a way to relate the properties for larger trees to the property for smaller trees.
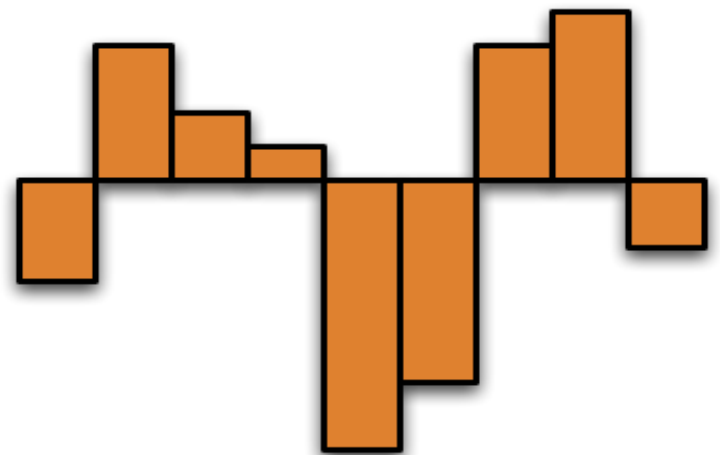
Introduction to Recurrence Relations

A technique for analyzing recursive algorithms

Recap: Maximum Contiguous Subsequence Sum problem

Problem definition: Given a non-empty sequence of n (possibly negative) integers $A_1, A_2, \ldots, A_n$, find the maximum consecutive subsequence $S_{i,j} = \sum_{k=i}^{j} A_k$, and the corresponding values of i and j.

Divide and Conquer Approach

▸ Split the sequence in half
▸ Where can the maximum subsequence appear?

▸ Three possibilities :
  ◦ entirely in the first half,
  ◦ entirely in the second half, or
  ◦ **begins** in the first half and **ends** in the second half
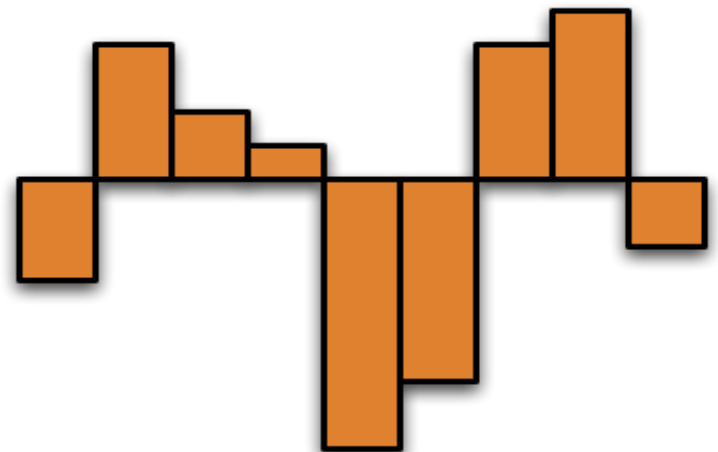
This leads to a recursive algorithm

1. Using recursion, find the maximum sum of **first** half of sequence
2. Using recursion, find the maximum sum of **second** half of sequence
3. Compute the max of all sums that begin in the first half and end in the second half
   ◦ (Use a couple of loops for this)
4. Choose the largest of these three numbers

```
private static int maxSumRec( int [ ] a, int left, int right )
{
    int maxLeftBorderSum = 0, maxRightBorderSum = 0;
    int leftBorderSum = 0, rightBorderSum = 0;
    int center = ( left + right ) / 2;

    if( left == right )  // Base case
        return a[ left ] > 0 ? a[ left ] : 0;

    int maxLeftSum  = maxSumRec( a, left, center );
    int maxRightSum = maxSumRec( a, center + 1, right );

    for( int i = center; i >= left; i-- )
    {
        leftBorderSum += a[ i ];
        if( leftBorderSum > maxLeftBorderSum )
            maxLeftBorderSum = leftBorderSum;
    }

    for( int i = center + 1; i <= right; i++ )
    {
        rightBorderSum += a[ i ];
        if( rightBorderSum > maxRightBorderSum )
            maxRightBorderSum = rightBorderSum;
    }

    return max3( maxLeftSum, maxRightSum,
                 maxLeftBorderSum + maxRightBorderSum );
```

N = array size

What's the run–time?

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Runtime =
Recursive part +
non–recursive part

Analysis?

▸ Write a **Recurrence Relation**

◦ T(N) gives the run-time as a function of N

◦ Two (or more) part definition:

- Base case,
  like T(1) = c

- Recursive case,
  like T(N) = T(N/2) + 1

So, what's the recurrence relation for the recursive MCSS algorithm?

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Recursive part +
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```

Runtime = Recursive part + non-recursive part

$T(N) = 2T(N/2) + \theta(N)$

$T(1) = 1$

Solve Simple Recurrence Relations

▸ One strategy: **guess and check**

▸ Examples:
As class:
◦ $T(0) = 0$, $T(N) = 2 + T(N-1)$
◦ $T(0) = 1$, $T(N) = 2\ T(N-1)$
◦ $T(0) = T(1) = 1$, $T(N) = T(N-2) + T(N-1)$

On quiz:
◦ $T(0) = 1$, $T(N) = N\ T(N-1)$
◦ $T(0) = 0$, $T(N) = T(N-1) + N$
◦ $T(1) = 1$, $T(N) = 2\ T(N/2) + N$
        (just consider the cases where $N=2^k$)

Next: Solution Strategies for Recurrence Relations

▸ Guess and check
▸ Substitution
▸ Telescoping and iteration
▸ The "master" method

▸ We'll see the last 3 next time

Editor Trees
Work Time

Please address any issues
found in Milestone 2 feedback