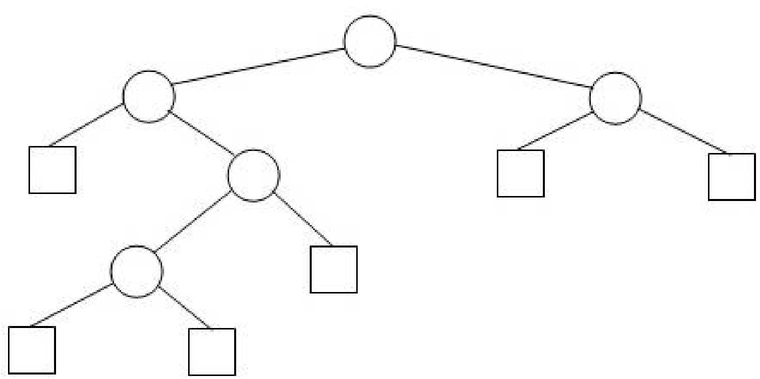
CSSE 230



Extended Binary Trees
Recurrence relations

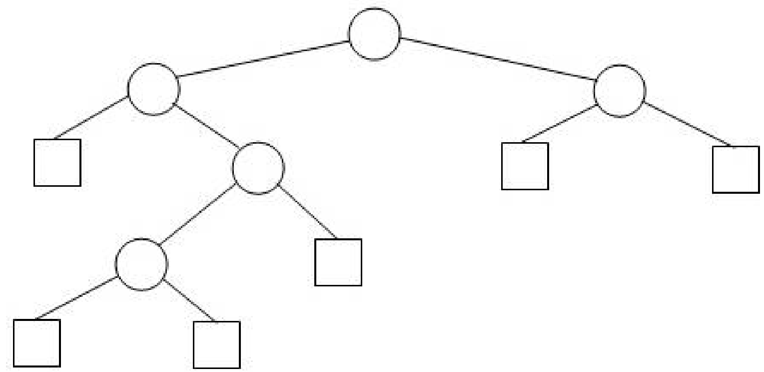
Reminders/Announcements

- Today:
  - Extended Binary Trees (basis for much of WA8, which includes induction proofs and no programming)
  - Recurrence relations, part 1
  - EditorTrees worktime?
- Next session: exam 2 (evening)

Extended Binary Trees (EBT's)

Bringing new life to Null nodes!

An Extended Binary Tree (EBT) just has *null* external nodes as leaves



- Not a single NULL_NODE, but many NULL_NODEs
- An Extended Binary tree is either
  - an ***external (null) node***, or
  - an (**internal**) root node and two EBTs T_L and T_R.
- We draw internal nodes as circles and external nodes as squares.
  - Generic picture and detailed picture.
- This is simply an alternative way of viewing binary trees, in which we view the external nodes as "places" where a search can end or an element can be inserted.

A property of EBTs

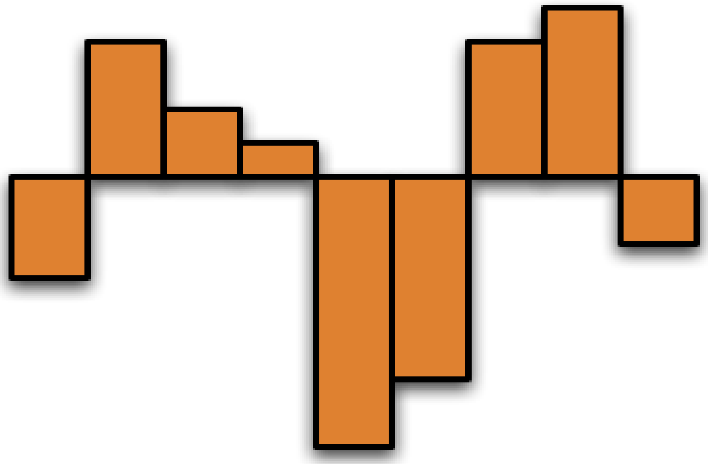
- **Property** P(N): For any N>=0, any EBT with N internal nodes has _______ external nodes.
- **Proof by strong induction**, based on the recursive definition.
  - A notation for this problem: IN(T), EN(T)
  - Note that, like a lot of other simple examples, this one can be done without induction.
  - But one purpose of this exercise is practice with strong induction, especially on binary trees.
- What is the crux of any induction proof?
  - Finding a way to relate the properties for larger values (in this case larger trees) to the property for smaller values (smaller trees). **Do the proof now**.

Introduction to Recurrence Relations

A technique for analyzing recursive algorithms

Recap: Maximum Contiguous Subsequence Sum problem

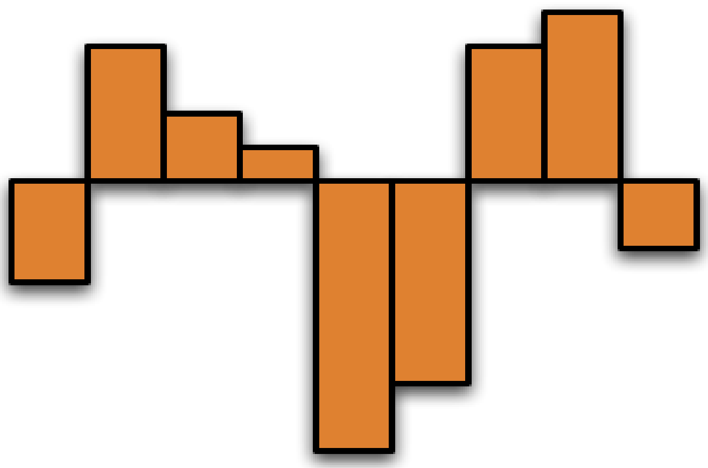
Problem definition: Given a non-empty sequence of n (possibly negative) integers $A_1, A_2, \ldots, A_n$, find the maximum consecutive subsequence $S_{i,j} = \sum_{k=i}^{j} A_k$, and the corresponding values of i and j.



Divide and Conquer Approach

- Split the sequence in half
- Where can the maximum subsequence appear?

- Three possibilities :
  - entirely in the first half,
  - entirely in the second half, or
  - **begins** in the first half and **ends** in the second half



This leads to a recursive algorithm

1. Using recursion, find the maximum sum of **first** half of sequence
2. Using recursion, find the maximum sum of **second** half of sequence
3. Compute the max of all sums that begin in the first half and end in the second half
    - (Use a couple of loops for this)
4. Choose the largest of these three numbers

```
private static int maxSumRec( int [ ] a, int left, int right )
{
    int maxLeftBorderSum = 0, maxRightBorderSum = 0;
    int leftBorderSum = 0, rightBorderSum = 0;
    int center = ( left + right ) / 2;

    if( left == right )  // Base case
        return a[ left ] > 0 ? a[ left ] : 0;

    int maxLeftSum  = maxSumRec( a, left, center );
    int maxRightSum = maxSumRec( a, center + 1, right );

    for( int i = center; i >= left; i-- )
    {
        leftBorderSum += a[ i ];
        if( leftBorderSum > maxLeftBorderSum )
            maxLeftBorderSum = leftBorderSum;
    }

    for( int i = center + 1; i <= right; i++ )
    {
        rightBorderSum += a[ i ];
        if( rightBorderSum > maxRightBorderSum )
            maxRightBorderSum = rightBorderSum;
    }

    return max3( maxLeftSum, maxRightSum,
                 maxLeftBorderSum + maxRightBorderSum );
```

So, what’s the run-time?

Analysis?

- Use a **Recurrence Relation**
  - Typically written T(N), gives the run-time as a function of N
  - Two (or more) part definition:
    - Base case, like T(1) = c
    - Recursive case, like T(N) = T(N/2)

So, what's the recurrence relation for the recursive MCSS algorithm?

```java
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```

What's N in the base case?

Recurrence Relation, Formally

- An equation (or inequality) that relates the n^{th} element of a sequence to certain of its predecessors (recursive case)
- Includes an initial condition (base case)
- **Solution**: A function of n.

- Similar to differential equation, but discrete instead of continuous
- Some solution techniques are similar to diff. eq. solution techniques

Solve Simple Recurrence Relations

- One strategy: **guess and check**

- Examples:
  - $T(0) = 0$, $T(N) = 2 + T(N-1)$
  - $T(0) = 1$, $T(N) = 2\,T(N-1)$
  - $T(0) = T(1) = 1$, $T(N) = T(N-2) + T(N-1)$
  - $T(0) = 1$, $T(N) = N\,T(N-1)$
  - $T(0) = 0$, $T(N) = T(N-1) + N$
  - $T(1) = 1$, $T(N) = 2\,T(N/2) + N$

    (just consider the cases where $N=2^k$)

Solution Strategies for Recurrence Relations

- Guess and check
- Substitution
- Telescoping and iteration
- The “master” method

- We’ll see the last 3 next time

Editor Trees Work Time