CSSE132
Introduction to Computer Systems

8 : Boolean Algebra

March 7, 2013

- Postulates
- Theorems
- DeMorgan’s Theorem
- Some Definitions
- Canonical Forms
- Canonical Forms of the Half Adder
- Complements and Conversions
- In-class Examples
- Basic Gates

P1: $A = 0$ if $A \neq 1$ $A=1$ if $A \neq 0$

P2: if $A = 0$ then $A'=1$ if $A = 1$ then $A'=0$

P3: $0*0=0$ $1+1=1$

P4: $1*1=1$ $0+0=0$

P5: $0*1=1*0=0$ $1+0=0+1=1$

Boolean Algebra: Theorems

T1: A+0=A A*1=A

T2: A+1=1 A*0=0

T3: A+A=A A*A=A

T4: (A')'=A

T5: A+A'=1 A*A'=0

T6: A+B=B+A A*B=B*A

T7: A+(B+C)=(A+B)+C
A*(B*C)=(A*B)*C

T8: A*B+A*C=A*(B+C)
(A+B)*(A+C)=A+B*C

T9: A+A*B=A A*(A+B)=A

T10: A*B+A*B'=A
(A+B)*(A+B')=A

T11: Demorgan's theorem
(A+B)'=A'*B'
(A*B)'=A'+B'

Proof of T10 by Perfect Induction

- Enumerate all combinations and show two expressions are identical.

$$\text{T10:}\quad A*B+A*B'=A$$

$$(A+B)*(A+B')=A$$

Proof of T10 by Perfect Induction

- Enumerate all combinations and show two expressions are identical.

$$T10: \; A*B+A*B' = A$$

$$(A+B)*(A+B') = A$$

A	B	A+B	A+B’	(A+B)(A+B’)	AB	AB’	AB+AB’
0	0	0	1	0	0	0	0
0	1	1	0	0	0	0	0
1	0	1	1	1	0	1	1
1	1	1	1	1	1	0	1

- Prove by perfect induction
- Enumerate all combinations and show two expressions are identical.

$$(A+B)' = A'B' \qquad \overline{(A+B)} = \overline{A} \bullet \overline{B}$$

$$(AB)' = A' + B' \qquad \overline{A \bullet B} = \overline{A} + \overline{B}$$

- Prove by perfect induction
- Enumerate all combinations and show two expressions are identical.

$(A+B)' = A'B'$

$(AB)' = A' + B'$

$$\overline{(A+B)} = \bar{A} \bullet \bar{B}$$

$$\overline{A \bullet B} = \bar{A} + \bar{B}$$

A	B	A'	B'	A+B	(A+B)'	A'B'	(AB)'	A'+B'
0	0	1	1	0	1	1	1	1
0	1	1	0	1	0	0	1	1
1	0	0	1	1	0	0	1	1
1	1	0	0	1	0	0	0	0

- Result : invert components, change operation!

Product term:	all variables are ANDed, A*B*C'*D
Sum term:	all variables are ORed, A+B+C'+D
Sum of products:	A*B+A*C*D
Product of sums:	(A+B)*(A+C')*(B+C+D)
Normal term:	a product or sum in which each variable appears no more than once
Minterm:	a normal product term containing all variables
Maxterm:	a normal sum term containing all variables

- Form one: sum of minterms
- Form two: product of maxterms
- Minterm number and Maxterm number

A	B	C	Minterm	Minterm number	Maxterm	Maxterm number
0	0	0	A′B′C′	∑(0)	A+B+C	Π(0)
0	0	1	A′B′C	∑(1)	A+B+C′	Π(1)
0	1	0	A′BC′	∑(2)	A+B′+C	Π(2)
0	1	1	A′BC	∑(3)	A+B′+C′	Π(3)
1	0	0	AB′C′	∑(4)	A′+B+C	Π(4)
1	0	1	AB′C	∑(5)	A′+B+C′	Π(5)
1	1	0	ABC′	∑(6)	A′+B′+C	Π(6)
1	1	1	A*B*C	∑(7)	A′+B′+C′	Π(7)

Example: Canonical Forms of the Half Adder

A	B	Sum	Carry
0	0	0	0
0	1	1	0
1	0	1	0
1	1	0	1

$Sum = A'B + AB' = \Sigma(1,2)$

$Sum = (A'B' + AB)' = (A+B)(A'+B') = \prod(3,0)$

$Carry = AB = \Sigma(3)$

$Carry = (A'B' + A'B + AB')' = \prod(0,1,2)$

- To complement a function, replace terms with those that are not present.
  - Complement of a function consists of all terms that causes “0”.
- To convert a function from the product form to the sum form, change the product symbol to the sum symbol and use the terms that are not present, vice versa.
  - Again, all those new terms form “0” entries in the truth table.

A	B	Sum	Carry
0	0	0	0
0	1	1	0
1	0	1	0
1	1	0	1

Complement of Sum:

$$Sum = \bar{A}B + A\bar{B} = \sum(1,2)$$

$$\overline{Sum} = \overline{\bar{A}B + A\bar{B}} = \sum(0,3)$$

Convert Sum from SOP to POS:

$$Sum = \bar{A}B + A\bar{B} = \sum(1,2)$$

$$= \prod(0,3) = (A+B)(\bar{A}+\bar{B})$$

- Commutative
  - $a + b = b + a$
  - $a * b = b * a$
- Distributive
  - $a * (b + c) = a * b + a * c$
  - $a + (b * c) = (a + b) * (a + c)$
- Associative
  - $(a + b) + c = a + (b + c)$
  - $(a * b) * c = a * (b * c)$
- Identity
  - $0 + a = a + 0 = a$
  - $1 * a = a * 1 = a$
- Complement
  - $a + a' = 1$
  - $a * a' = 0$
- To prove, just evaluate all possibilities.

Example Applications of Boolean Algebra Properties

- Show abc' equivalent to c'ba.
  - Use commutative property:
    a*b*c' = a*c'*b = c'*a*b = c'*b*a
- Show abc + abc' = ab.
  - Use first distributive property
    abc + abc' = ab(c+c').
  - Complement property
    Replace c+c' by 1: ab(c+c') = ab(1).
  - Identity property
    ab(1) = ab*1 = ab.
- Show x + x'z equivalent to x + z.
  - Second distributive property
    Replace x+x'z by (x+x')*(x+z).
  - Complement property
    Replace (x+x') by 1,
  - Identity property
    replace 1*(x+z) by x+z.

1. ***Simplify the expression with Boolean algebra and indicate which theorems are used.***

$$Z(A,B,C) = A \bullet B \bullet \overline{C} + \overline{A} \bullet B + \overline{A} \bullet \overline{B} \bullet C$$

2. ***Complement the following expression, using DeMorgan's theorem to produce a product of sums expression.***

$$Z(A,B,C) = A \bullet B \bullet \overline{C} + A \bullet C$$

3. ***Obtain the simplified logic expression for the majority voting function.***

A	B	C	F
0	0	0	
0	0	1	
0	1	0	
0	1	1	
1	0	0	
1	0	1	
1	1	0	
1	1	1	

Solution for Problem 1 of In-class Exercise

- ***Simplify with Boolean algebra and indicate which theorems are used.***

$$Z(A,B,C) = A \bullet B \bullet \overline{C} + \overline{A} \bullet B + \overline{A} \bullet \overline{B} \bullet C$$

$$= A \bullet B \bullet \overline{C} + \overline{A} \bullet B \bullet (C + \overline{C}) + \overline{A} \bullet \overline{B} \bullet C \quad \text{T5, T1}$$

$$= A \bullet B \bullet \overline{C} + \overline{A} \bullet B \bullet C + \overline{A} \bullet B \bullet \overline{C} + \overline{A} \bullet \overline{B} \bullet C \quad \text{T8}$$

$$= (A + \overline{A})B \bullet \overline{C} + \overline{A} \bullet C(B + \overline{B}) \quad \text{T8}$$

$$= B \bullet \overline{C} + \overline{A} \bullet C \quad \text{T10}$$

Solution for Problem 2 of In-class Exercise

- ***Complement the following, using DeMorgan's theorem to produce a product of sums expression.***

$$Z(A,B,C) = A \bullet B \bullet \overline{C} + A \bullet C$$

$$\overline{Z}(A,B,C) = \overline{A \bullet B \bullet \overline{C} + A \bullet C} = \overline{A \bullet B \bullet \overline{C}} \bullet \overline{A \bullet C}$$

$$= (\overline{A} + \overline{B} + C)(\overline{A} + \overline{C})$$

$$Z(A,B,C) = \overline{(\overline{A} + \overline{B} + C)(\overline{A} + \overline{C})}$$

Solution for Problem 3 of In-class Exercise

- ***Obtain simplified logic expression for the majority voting function.***

$$\bar{A} \bullet B \bullet C + A \bullet \bar{B} \bullet C + A \bullet B \bullet \bar{C} + A \bullet B \bullet C$$

$$= \bar{A} \bullet B \bullet C + A \bullet B \bullet C + A \bullet \bar{B} \bullet C + A \bullet B \bullet \bar{C} + A \bullet B \bullet C + A \bullet B \bullet C \quad \text{T3}$$

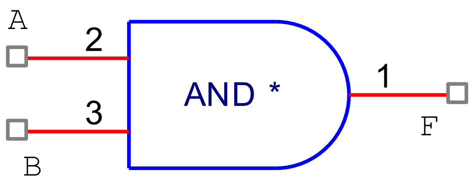
$$= BC + AC + AB \quad \text{T8, T5, T1}$$

A	B	C	F
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

Basic Gates: AND Gate, OR Gate and NOT Gate (Inverter)

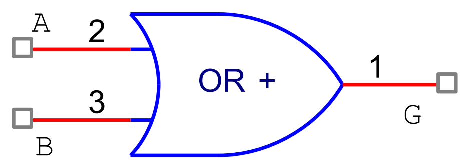
A	B	F
0	0	0
0	1	0
1	0	0
1	1	1

A	B	G
0	0	0
0	1	1
1	0	1
1	1	1



AND Gate

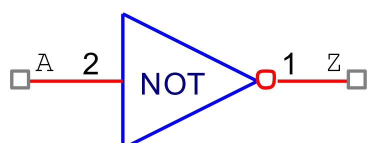
$$F = A * B$$



OR Gate

$$F = A + B$$

A	z
0	1
1	0



Inverter Gate

$$Z = A' = \overline{A}$$

- Combine AND, OR, NOT to form new gates
  - NAND : NOT(AND)
  - NOR : NOT(OR)
  - XOR : $A \oplus B = A * B' + A' * B$
  - XNOR : NOT(XOR)

- NAND and NOR can be built without AND or OR gates
  - All gates can be implemented with NAND and NOR
  - Called ‘Universal logic gate’