Steady-State, 2-D Diffusion Example
Linear Triangles

Having applied the GWS "recipe" for the two-dimensional steady diffusion equation,

$$\begin{aligned} \mathscr{L}\big(T(x,y)\big) &= -\frac{\partial}{\partial x}\left(k\frac{\partial T(x,y)}{\partial x}\right) - \frac{\partial}{\partial y}\left(k\frac{\partial T(x,y)}{\partial y}\right) - s(x,y) = 0 \\ &\equiv -\nabla\cdot\big(k\nabla T(x,y)\big) - s(x,y) = 0 \qquad on\ \Omega \subset \Re^2 \\ l(T) &= -k\nabla T\cdot\hat{n} - q_n = 0 \qquad on\ \partial\Omega_1 \\ T(x,y)_b &= T_b \qquad on\ \partial\Omega_2 \end{aligned} \tag{1}$$

we obtained

$$GWS^h = S_e\left(\int_{\Omega_e}\nabla\{N_k\}\cdot\nabla\{N_k\}^T d\Omega_e\{Q\}_e - \int_{\Omega_e}\{N_k\}\{N_k\}^T d\Omega_e\{S\}_e + \int_{\partial\Omega_1}\{N_k\}\{N_k\}^T d\sigma_e\{QN\}_e = \{0\}_e\right) \tag{2}$$

where, for linear triangles and constant thermal conductivity, the diffusion master matrix $[\mathrm{DIFF}]_e$ is

$$[DIFF]_e = \int_{\Omega_e}\nabla\{N_1\}\cdot k\nabla\{N_1\}^T d\Omega_e = \frac{k}{4A_e}\begin{bmatrix}\zeta_{11}^2+\zeta_{12}^2 & \zeta_{11}\zeta_{21}+\zeta_{12}\zeta_{22} & \zeta_{11}\zeta_{31}+\zeta_{12}\zeta_{32} \\ & \zeta_{21}^2+\zeta_{22}^2 & \zeta_{21}\zeta_{31}+\zeta_{22}\zeta_{32} \\ sym & & \zeta_{31}^2+\zeta_{32}^2\end{bmatrix}_e \tag{3}$$

and associated entries of the coordinate transformation jacobian $[\mathrm{J}]_e$ are

$$\begin{bmatrix}\zeta_{11} & \zeta_{12} \\ \zeta_{21} & \zeta_{22} \\ \zeta_{31} & \zeta_{32}\end{bmatrix}_e = \begin{bmatrix}Y2-Y3 & X3-X2 \\ Y3-Y1 & X1-X3 \\ Y1-Y2 & X2-X1\end{bmatrix}_e \tag{4}$$

The source master matrix $\{\mathrm{SRC}\}_e$ is

$$\{SRC\}_e = \int_{\Omega_e}\{N_k\}\{N_k\}^T d\Omega_e\{S\}_e = \frac{A_e}{12}\begin{bmatrix}2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2\end{bmatrix}\{S\}_e \tag{5}$$

The applied normal boundary flux $\{\mathrm{BFLX}\}_e$, interpolated over the one-dimensional boundary edge between nodes A and B, has a length scale of

$$l_e = \sqrt{(XB-XA)^2 + (YB-YA)^2}$$

thus

$$\{BFLX\}_e = \int_{\partial\Omega_1}\{N_k\}\{N_k\}^T d\sigma\{QN\}_e = \int_{l_e}\{N_k\}\{N_k\}^T d\bar{x}\{QN\}_e = \int_{l_e}\begin{Bmatrix}\zeta_A \\ \zeta_B \\ 0\end{Bmatrix}\{\zeta_A \quad \zeta_B \quad 0\}d\bar{x}\begin{Bmatrix}QNA \\ QNB \\ 0\end{Bmatrix}_e \tag{6}$$

where (6) will yield a 3x3 matrix similar to

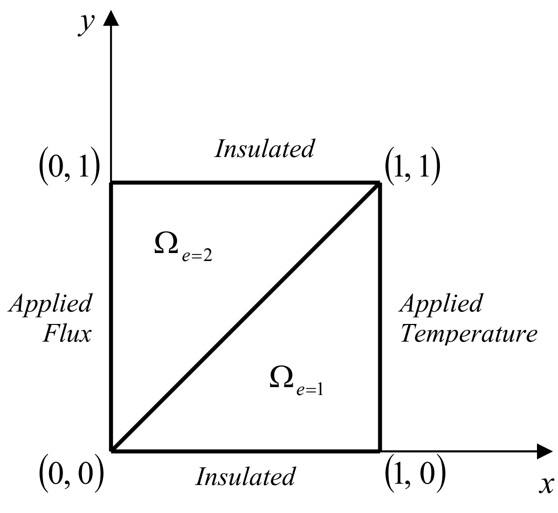
$$\{BFLX\}_e = \int_{l_e} \begin{Bmatrix} \zeta_A \\ \zeta_B \\ 0 \end{Bmatrix} \{\zeta_A \quad \zeta_B \quad 0\} d\bar{x} \{QN\}_e = \frac{l_e}{6} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} QNA \\ QNB \\ 0 \end{Bmatrix}_e \tag{7}$$

except that the zeros will occur in the row and column numbers equal to the node that is not on the boundary. Armed with the above definitions, we are ready to assemble and solve a two-element discretization of an example problem.

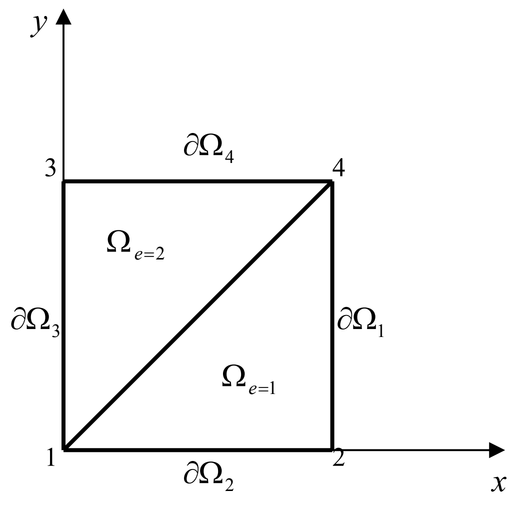
Example: Determine the steady state temperature distribution in a square slab on $\Omega = (0 < x < 1, 0 < y < 1)$ subject to the boundary conditions

$$T(x=1, y) = 100 \quad on\ \partial\Omega_1 \qquad \frac{\partial T}{\partial n}(x=0, y) = -1 \quad on\ \partial\Omega_3$$
$$\frac{\partial T}{\partial n}(x, y=0) = 0 \quad on\ \partial\Omega_2 \qquad \frac{\partial T}{\partial n}(x, y=1) = 0 \quad on\ \partial\Omega_4$$

The slab is homogenous with a constant thermal conductivity of k=2 and no source.



Global Coordinates and BC's



Node Numbering and Boundaries

Element 1 - $\Omega_{e=1}$:

Local Node	Global Node	Global X	Global Y
1	*1*	0	0
2	*2*	1	0
3	*4*	1	1

$$\begin{bmatrix}\zeta_{11} & \zeta_{12}\\ \zeta_{21} & \zeta_{22}\\ \zeta_{31} & \zeta_{32}\end{bmatrix}_{e=1} = \begin{bmatrix}Y2-Y3 & X3-X2\\ Y3-Y1 & X1-X3\\ Y1-Y2 & X2-X1\end{bmatrix}_{e=1} = \begin{bmatrix}-1 & 0\\ 1 & -1\\ 0 & 1\end{bmatrix}_{e=1}$$

$$A_{e=1} = 0.5 \qquad l^{1,2}_{e=1} = 1 \qquad l^{2,3}_{e=1} = 1$$

Evaluating $[\mathrm{DIFF}]_{e=1}$:

$$[DIFF]_{e=1} = \frac{k}{4A_{e=1}}\begin{bmatrix}\zeta_{11}^2+\zeta_{12}^2 & \zeta_{11}\zeta_{21}+\zeta_{12}\zeta_{22} & \zeta_{11}\zeta_{31}+\zeta_{12}\zeta_{32}\\ & \zeta_{21}^2+\zeta_{22}^2 & \zeta_{21}\zeta_{31}+\zeta_{22}\zeta_{32}\\ sym & & \zeta_{31}^2+\zeta_{32}^2\end{bmatrix}_{e=1} = 1\begin{bmatrix}1 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 1\end{bmatrix}_{e=1}$$

Expressing the vector of unknowns $\{Q\}_{e=1}$ in terms of the *global* numbering:

$$\{Q\}_{e=1} = \begin{Bmatrix}Q1\\ Q2\\ Q4\end{Bmatrix}_{e=1}$$

Evaluating $\{\mathrm{BFLX}\}_{e=1}$ on edge 1,2:

$$\{BFLX\}^{1,2}_{e=1} = \int_{l_e}\begin{Bmatrix}\zeta_A\\ \zeta_B\\ 0\end{Bmatrix}\{\zeta_A \quad \zeta_B\}d\overline{x}\begin{Bmatrix}QNA\\ QNB\\ 0\end{Bmatrix}_{e=1} = \frac{l^{1,2}_{e=1}}{6}\begin{bmatrix}2 & 1 & 0\\ 1 & 2 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{Bmatrix}0\\ 0\\ 0\end{Bmatrix}_{e=1} = \frac{1}{6}\begin{Bmatrix}0\\ 0\\ 0\end{Bmatrix}_{e=1}$$

Note that, since a Dirichlet temperature is applied to boundary edge 2,3, no vector needs be formed.

Element 2 - $\Omega_{e=2}$:

Local Node	Global Node	Global X	Global Y
1	*1*	0	0
2	*4*	1	1
3	*3*	0	1

$$\begin{bmatrix} \zeta_{11} & \zeta_{12} \\ \zeta_{21} & \zeta_{22} \\ \zeta_{31} & \zeta_{32} \end{bmatrix}_{e=2} = \begin{bmatrix} Y2-Y3 & X3-X2 \\ Y3-Y1 & X1-X3 \\ Y1-Y2 & X2-X1 \end{bmatrix}_{e=2} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ -1 & 1 \end{bmatrix}_{e=2}$$

$$A_{e=2} = 0.5 \qquad l_{e=2}^{2,3} = 1 \qquad l_{e=2}^{3,1} = 1$$

Evaluating [DIFF]$_{e=2}$:

$$[DIFF]_{e=2} = \frac{k}{4A_{e=2}} \begin{bmatrix} \zeta_{11}^2 + \zeta_{12}^2 & \zeta_{11}\zeta_{21} + \zeta_{12}\zeta_{22} & \zeta_{11}\zeta_{31} + \zeta_{12}\zeta_{32} \\ & \zeta_{21}^2 + \zeta_{22}^2 & \zeta_{21}\zeta_{31} + \zeta_{22}\zeta_{32} \\ sym & & \zeta_{31}^2 + \zeta_{32}^2 \end{bmatrix}_{e=2} = 1 \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 2 \end{bmatrix}_{e=2}$$

Expressing the vector of unknowns {Q}$_{e=2}$ in terms of the *global* numbering:

$$\{Q\}_{e=2} = \begin{Bmatrix} Q1 \\ Q4 \\ Q3 \end{Bmatrix}_{e=2}$$

Evaluating {BFLX}$_{e=2}$ on edge 2,3:

$$\{BFLX\}_{e=2}^{2,3} = \int_{l_e} \begin{Bmatrix} 0 \\ \zeta_A \\ \zeta_B \end{Bmatrix} \{0 \quad \zeta_A \quad \zeta_B\} d\bar{x} \begin{Bmatrix} 0 \\ QNA \\ QNB \end{Bmatrix}_{e=2} = \frac{l_{e=2}^{2,3}}{6} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}_{e=2} = \frac{1}{6} \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}_{e=2}$$

Evaluating {BFLX}$_{e=2}$ on edge 3,1:

$$\{BFLX\}_{e=2}^{3,1} = \int_{l_e} \begin{Bmatrix} \zeta_A \\ 0 \\ \zeta_B \end{Bmatrix} \{\zeta_A \quad 0 \quad \zeta_B\} d\bar{x} \begin{Bmatrix} QNA \\ 0 \\ QNB \end{Bmatrix}_{e=2} = \frac{l_{e=2}^{3,1}}{6} \begin{bmatrix} 2 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{Bmatrix} -1 \\ 0 \\ -1 \end{Bmatrix}_{e=2} = \frac{1}{6} \begin{Bmatrix} -3 \\ 0 \\ -3 \end{Bmatrix}_{e=2}$$

Assembly:

To assemble each element level matrix into the global matrix requires building the “padded” matrix statements.

For element 1:

$$[DIFF]_{e=1}\{Q\}_{e=1} + \{BFLX\}_{e=1} = \{0\}_{e=1}$$

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{Bmatrix} Q1 \\ Q2 \\ Q3 \\ Q4 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

For element 2:

$$[DIFF]_{e=2}\{Q\}_{e=2} + \{BFLX\}_{e=2} = \{0\}_{e=2}$$

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} Q1 \\ Q2 \\ Q3 \\ Q4 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} -0.5 \\ 0 \\ -0.5 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Adding the padded element matrices to form the global matrix statement:

$$\begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix} \begin{Bmatrix} Q1 \\ Q2 \\ Q3 \\ Q4 \end{Bmatrix} = \begin{Bmatrix} 0.5 \\ 0 \\ 0.5 \\ 0 \end{Bmatrix}$$

Applying the fixed temperature at nodes 2 and 4 yields the terminal form of the matrix statement

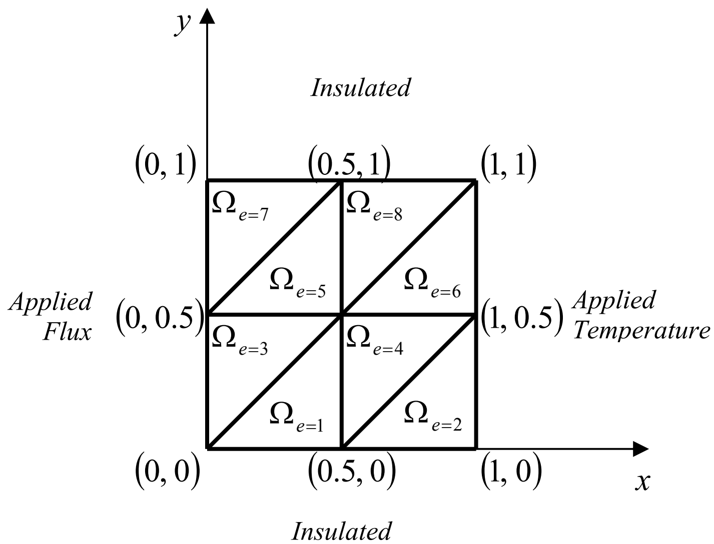
$$\begin{bmatrix} 2 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 2 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} Q1 \\ Q2 \\ Q3 \\ Q4 \end{Bmatrix} = \begin{Bmatrix} 0.5 \\ 100 \\ 0.5 \\ 100 \end{Bmatrix}$$

which is readily solved to give:

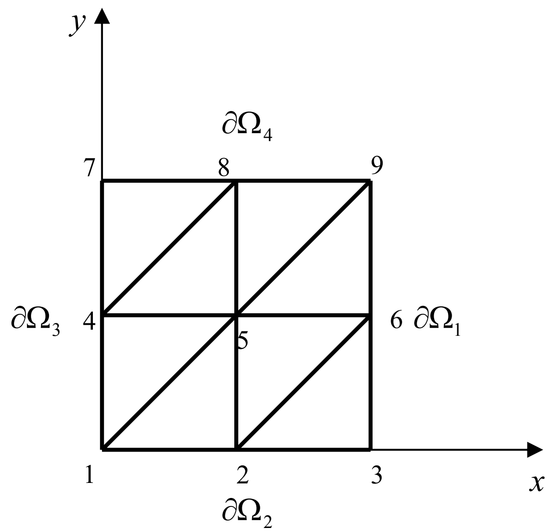
$$Q1 = 100.5 \quad Q2 = 100 \quad Q3 = 100.5 \quad Q4 = 100$$

Homework:

For the same domain and boundary conditions, repeat the above for the uniformly refined mesh given below. How does your solution differ from that obtained in class?



Global Coordinates and BC's



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