Linear Basis Functions
Triangles and Tetrahedrons

Previously we introduced the one-dimensional linear basis function $N_1(\zeta_i)$ as a tool for interpolating our unknown solution over an element via

$$T_e(x) = \zeta_1 Q1_e + \zeta_2 Q2_e = \{\zeta_1 \ \zeta_2\}\begin{Bmatrix} Q1 \\ Q2 \end{Bmatrix}_e = \{N_1(\zeta_i)\}^T \{Q\}_e$$

where

$$\zeta_1 = \frac{X2 - x}{l_e} \qquad \zeta_2 = \frac{-X1 + x}{l_e} \qquad l_e = X2 - X1 \tag{1}$$

An important property of basis functions is *nodal consistency*: evaluating the basis function at its associated node must yield a 1 while evaluating it any other nodal location must yield a zero. As an example, evaluating ζ_1 from (1) at X1 yields a 1 while evaluating it at X2 yields a zero. Recall that this idea of nodal consistency was how we constructed the quadratic and cubic basis functions as well.

Let's rederive (1) in order to set the stage for going multi-dimensional. We want to generate a one-dimensional (*n*=1) linear approximation to a function T(x) over an element with nodal coordinates of X1 and X2. We can express it generically as

$$T_e(x) = a_1 + a_2 x \tag{2}$$

At the left side (node 1, coordinate X1) it will have the value of Q1 and at the right side (node 2, coordinate X2) it will have the value of Q2. Thus:

$$\begin{aligned} T_e(x = X1_e) = Q1_e \quad &\Rightarrow \quad Q1_e = a_1 + a_2 X1_e \\ Te(x = X2_e) = Q2_e \quad &\Rightarrow \quad Q2_e = a_1 + a_2 X2_e \end{aligned} \tag{3}$$

Equation (3) can be rewritten as a matrix statement

$$\begin{bmatrix} 1 & X1 \\ 1 & X2 \end{bmatrix}_e \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} Q1 \\ Q2 \end{Bmatrix}_e \tag{4}$$

which can be solved for a_1 and a_2

$$a_1 = \frac{(X2Q1 - X1Q2)_e}{l_e} \qquad a_2 = \frac{(Q2 - Q1)_e}{l_e} \tag{5}$$

where l_e is the element length and is computed as the determinant of the element coordinate matrix in (4).

$$\det\begin{bmatrix} 1 & X1 \\ 1 & X2 \end{bmatrix}_e = (X2 - X1)_e = l_e$$

Substituting (5) into (2)

$$T_e(x) = \left(\frac{(X2Q1 - X1Q2)_e}{l_e}\right) + \left(\frac{(Q2 - Q1)_e}{l_e}\right)x \tag{6}$$

Recognizing that (5) contains the nodal values of T $(Q1,Q2)_e$ as common factors, some algebra reveals

$$T_e(x) = \left(\frac{X2_e - x}{l_e}\right)Q1_e + \left(\frac{-X1_e + x}{l_e}\right)Q2_e \tag{7}$$

and the basis functions are readily revealed and correspond exactly to (1)

Now, how about two-dimensional triangles? Since a triangle has three vertex nodes, the linear triangular element which will interpolate our function can be expressed as

$$T_e(x) = \zeta_1 Q1_e + \zeta_2 Q2_e + \zeta_3 Q3_e = \{\zeta_1 \quad \zeta_2 \quad \zeta_3\}\begin{Bmatrix} Q1 \\ Q2 \\ Q3 \end{Bmatrix}_e = \{N_1(\zeta_i)\}^T\{Q\}_e$$

The generic linear interpolation of $T_e(x,y)$ in two dimensions (n=2) is given as

$$T_e(x,y) = a_1 + a_2 x + a_3 y \tag{8}$$

Evaluating (8) at each of the vertex nodes

$$\begin{aligned} T_e(x = X1_e, y = Y1_e) = Q1_e &\quad\Rightarrow\quad Q1_e = a_1 + a_2 X1_e + a_3 Y1_e \\ T_e(x = X2_e, y = Y2_e) = Q2_e &\quad\Rightarrow\quad Q2_e = a_1 + a_2 X2_e + a_3 Y2_e \\ T_e(x = X3_e, y = Y3_e) = Q3_e &\quad\Rightarrow\quad Q3_e = a_1 + a_2 X3_e + a_3 Y3_e \end{aligned} \tag{9}$$

which can be written in matrix form as

$$\begin{bmatrix} 1 & X1 & Y1 \\ 1 & X2 & Y2 \\ 1 & X3 & Y3 \end{bmatrix}_e \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{Bmatrix} Q1 \\ Q2 \\ Q3 \end{Bmatrix}_e \tag{10}$$

which can be solved for a_1, a_2, and a_3

$$\begin{aligned} a_1 &= \frac{(X2Y3 - X3Y2)Q1 + (X3Y1 - X1Y3)Q2 + (X1Y2 - X2Y1)Q3}{2A_e} \\ a_2 &= \frac{(Y2 - Y3)Q1 + (Y3 - Y1)Q2 + (Y1 - Y2)Q3}{2A_e} \\ a_3 &= \frac{(X3 - X2)Q1 + (X1 - X3)Q2 + (X2 - X1)Q3}{2A_e} \end{aligned} \tag{11}$$

where $2A_e$ is twice the plane area of the triangular element and is computed as the determinant of the element coordinate matrix in (10)

$$\det[\ ] = (X1Y2 - X2Y1) + (X3Y1 - X1Y3) + (X2Y3 - X3Y2) = 2A_e$$

Substituting (11) into (8) and grouping by the common factors of Q reveals the basis functions to be

$$\begin{aligned} \zeta_1 &= \frac{(X2Y3 - X3Y2) + (Y2 - Y3)x + (X3 - X2)y}{2A_e} \\ \zeta_2 &= \frac{(X3Y1 - X1Y3) + (Y3 - Y1)x + (X1 - X3)y}{2A_e} \\ \zeta_3 &= \frac{(X1Y2 - X2Y1) + (Y1 - Y2)x + (X2 - X1)y}{2A_e} \end{aligned} \tag{12}$$

The official name given to the basis functions is the *natural coordinate system* as it uses global coordinates to “naturally” define the bases.

Homework:

1. Verify nodal consistency for the basis function for the two-dimensional linear triangle element.

2. Derive the basis function for the three-dimensional (n=3) linear tetrahedral element.
   Hint: the determinant of the element coordinate matrix is equal to $6V_e$

3. Verify nodal consistency for the basis function of the three-dimensional tetrahedral element.

4. What is the relationship between the determinant of the coordinate matrix and the dimension of the element?