Open Channel Flow II

Having found that approaching open channel flow (or any coupled PDE system involving non-linearities within the kinetic flux vector!) from the primitive variable perspective resulted in a decoupled Newton algorithm that would converge linearly at best and diverge at worst. Let us now employ the non-divergence form of the problem statement and see where we can go.

$$\mathsf{L}(\overline{q}) = \frac{\partial \overline{q}}{\partial t} + \overline{A}\frac{\partial \overline{q}}{\partial x} = 0 \qquad on\ \Omega \tag{1a}$$

where

$$\overline{q} = \begin{Bmatrix} h \\ m \end{Bmatrix} \quad and \quad \overline{A} = \frac{\partial f(\overline{q})}{\partial \overline{q}} = \begin{bmatrix} 0 & 1 \\ -\dfrac{m^2}{h^2} + gh & 2\dfrac{m}{h} \end{bmatrix} \tag{1b}$$

Substituting (1b) into (1a) and expanding

$$\begin{aligned} \mathsf{L}(h) &= \frac{\partial h}{\partial t} + 0\frac{\partial h}{\partial x} + 1\frac{\partial m}{\partial x} = 0 \qquad on\ \Omega \\ \mathsf{L}(m) &= \frac{\partial m}{\partial t} + \left(-\frac{m^2}{h^2} + gh\right)\frac{\partial h}{\partial x} + \left(2\frac{m}{h}\right)\frac{\partial m}{\partial x} = 0 \qquad on\ \Omega \end{aligned} \tag{2}$$

Assuming a series approximation to the unknown state variables and associated grouped variables

$$\begin{aligned} h(t,x) &\approx h^N(t,x) = \Psi_\alpha(x) H_\alpha(t) \qquad for \quad 1 \le \alpha \le N \\ m(t,x) &\approx m^N(t,x) = \Psi_\alpha(x) M_\alpha(t) \qquad for \quad 1 \le \alpha \le N \\ \frac{m^2}{h^2}(t,x) &\approx \left(\frac{m^2}{h^2}\right)^N(t,x) = \Psi_\alpha(x) MSHS_\alpha(t) \qquad for \quad 1 \le \alpha \le N \\ \frac{m}{h}(t,x) &\approx \left(\frac{m}{h}\right)^N(t,x) = \Psi_\alpha(x) MH_\alpha(t) \qquad for \quad 1 \le \alpha \le N \end{aligned} \tag{3}$$

Forming the Galerkin Weak Statement

$$\begin{aligned} GWS_h^N &= \int_\Omega \Psi_\beta \left(\frac{\partial h^N}{\partial t} + \frac{\partial m^N}{\partial x}\right) dx = 0 \\ GWS_m^N &= \int_\Omega \Psi_\beta \left(\frac{\partial m^N}{\partial t} + \left(-\left(\frac{m^2}{h^2}\right)^N + gh^N\right)\frac{\partial h^N}{\partial x} + 2\left(\frac{m}{h}\right)^N \frac{\partial m^N}{\partial x}\right) dx = 0 \end{aligned} \tag{4}$$

Substituting the series expansions and expanding

$$
\begin{aligned}
GWS_H^N &= \int_\Omega \Psi_\beta \Psi_\alpha dx \frac{dH_\alpha}{dt} + \int_\Omega \Psi_\beta \frac{d\Psi_\alpha}{dx} dx M_\alpha = 0 \\
GWS_M^N &= \int_\Omega \Psi_\beta \Psi_\alpha dx \frac{dM_\alpha}{dt} - MSHS_\alpha \int_\Omega \Psi_\alpha \Psi_\beta \frac{d\Psi_\alpha}{dx} dx H_\alpha \\
&\quad + gH_\alpha \int_\Omega \Psi_\alpha \Psi_\beta \frac{d\Psi_\alpha}{dx} dx H_\alpha + 2MH_\alpha \int_\Omega \Psi_\alpha \Psi_\beta \frac{d\Psi_\alpha}{dx} dx M_\alpha = 0
\end{aligned}
\tag{5}
$$

Discretizing

$$
\begin{aligned}
GWS_H^h &= S_e\left(\int_{\Omega_e} \{N_k\}\{N_k\}^T d\bar{x} \frac{d\{H\}_e}{dt} + \int_{\Omega_e} \{N_k\} \frac{d\{N_k\}^T}{dx} d\bar{x} \{M\}_e = \{0\}_e \right) \\
GWS_M^h &= S_e\Bigg(\int_{\Omega_e} \{N_k\}\{N_k\}^T d\bar{x} \frac{d\{M\}_e}{dt} - \{MSHS\}_e^T \int_{\Omega_e} \{N_k\}\{N_k\} \frac{d\{N_k\}^T}{dx} d\bar{x} \{H\}_e \\
&\quad + g\{H\}_e^T \int_{\Omega_e} \{N_k\}\{N_k\} \frac{d\{N_k\}^T}{dx} d\bar{x} \{H\}_e + 2\{MH\}_e^T \int_{\Omega_e} \{N_k\}\{N_k\} \frac{d\{N_k\}^T}{dx} d\bar{x} \{M\}_e = \{0\}_e \Bigg)
\end{aligned}
\tag{6}
$$

Upon assembly, the spatially discretized form (6) yields two coupled ordinary differential equations in time. Identifying the usual master matrices

$$
\begin{aligned}
GWS_H^h &= [\mathrm{MASS}] \frac{d\{H\}}{dt} + [\mathrm{CONVM1}]\{M\} = \{0\} \\
GWS_M^h &= [\mathrm{MASS}] \frac{d\{M\}}{dt} + [\mathrm{CONVH1}]\{H\} + [\mathrm{CONVH2}]\{H\} + [\mathrm{CONVM2}]\{M\} = \{0\}
\end{aligned}
\tag{7}
$$

Unlike the previous formulation, we are able to combine the two equations in (7) into one large matrix statement

$$
\begin{array}{lcccccccc}
[\mathrm{MASS}] \frac{d\{H\}}{dt} & + & & & & & [\mathrm{CONVM1}]\{M\} & = & \{0\} \\
[\mathrm{MASS}] \frac{d\{M\}}{dt} & + & [\mathrm{CONVH1}]\{H\} & + & [\mathrm{CONVH2}]\{H\} & + & [\mathrm{CONVM2}]\{M\} & = & \{0\}
\end{array}
$$

$$
\begin{bmatrix} MASS & 0 \\ 0 & MASS \end{bmatrix} \frac{d}{dt} \begin{Bmatrix} H \\ M \end{Bmatrix} + \begin{bmatrix} 0 & \mathrm{CONVM1} \\ \mathrm{CONVH1} + \mathrm{CONVH2} & \mathrm{CONVM2} \end{bmatrix} \begin{Bmatrix} H \\ M \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}
$$

which is now of the form

$$
[\mathrm{BIGMASS}] \frac{d\{Q\}}{dt} + [\mathrm{BIGCONV}]\{Q\} = \{0\} \tag{8}
$$

Employing the Theta Taylor Series to move the solution forward in time

$$
\Theta\mathrm{TS} = [\mathrm{BIGMASS}]\{\Delta Q\}_{n,n+1} + \Delta t \left(\Theta\{\mathrm{BIGRES}\}_{n+1} + (1-\Theta)\{\mathrm{BIGRES}\}_n \right) = \{0\} \tag{9}
$$

To solve our big non-linear equation, we shall call upon Newton

$$\{FQ\}_{n,n+1}^{p} = \Theta\mathrm{TS}_{\mathrm{n,n+1}}^{p} = [\mathrm{BIGMASS}]\{\Delta Q\}_{n,n+1}^{p} + \Delta t\left(\Theta\{\mathrm{BIGRES}\}_{n+1}^{p} + (1-\Theta)\{\mathrm{BIGRES}\}_{n}\right)$$

The total residual was easy - now for the jacobian.

$$[JAC]_{n+1}^{p} = \frac{\partial\{FQ\}_{n,n+1}^{p}}{\partial\{Q\}_{n+1}}$$

The temporal term is readily evaluated as

$$\frac{\partial}{\partial\{Q\}_{n+1}}\left([\mathrm{BIGMASS}]\{\Delta Q\}_{n,n+1}\right) = [\mathrm{BIGMASS}][BIG1]$$

Evaluating the spatial jacobian

$$\frac{\partial}{\partial\{Q\}_{n+1}}\left(\Delta t\left(\Theta\{\mathrm{BIGRES}\}_{n+1}^{p} + (1-\Theta)\{\mathrm{BIGRES}\}_{n}\right)\right) = \Delta t\Theta\frac{\partial}{\partial\{Q\}_{n+1}}\{\mathrm{BIGRES}\}_{n+1}^{p}$$

Taking our cue from the temporal development, the spatial jacobian will consist of four blocks

$$\begin{bmatrix} JHH & JHM \\ JMH & JMM \end{bmatrix}$$