The Peclet Problem

The convection-diffusion equation

$$
\begin{aligned}
&\mathsf{L}(T) = \frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} - \frac{\partial}{\partial x}\left(\frac{k}{\rho c_p}\frac{\partial T}{\partial x}\right) - \frac{s}{\rho c_p} = 0 && \Omega \in (0, L)\\
&\mathsf{l}(T(x_L, t)) = k\frac{\partial T}{\partial x} + h(T - T_r) = 0 && \partial\Omega_1 \in 0\\
&T(x_R, t) = T_R && \partial\Omega_2 \in L\\
&T(x, t_o) = T_o(x) && \Omega, \partial\Omega
\end{aligned}
\tag{1}
$$

can be further explored upon non-dimensionalization. Letting a superscript asterisk denote a non-dimensional variable, the non-dimensionalization of (1) involves the following definitions:

$$
\begin{aligned}
x^* &= \frac{1}{L}x & k^* &= \frac{1}{k_{ref}}k\\
t^* &= \frac{U}{L}t & \rho^* &= \frac{1}{\rho_{ref}}\rho\\
T^* &= \frac{T - T_{\min}}{T_{\max} - T_{\min}} & c_p^* &= \frac{1}{c_{p,ref}}c_p\\
u^* &= \frac{1}{U}u & h^* &= \frac{L}{k_{ref}}h \equiv Nu\\
s^* &= \frac{L}{U(T_{\max} - T_{\min})\rho_{ref}c_{p,ref}}s
\end{aligned}
\tag{2}
$$

Substituting the above definitions along with the non-d groups $\mathrm{Re} = \rho_{ref}UL/\mu_{ref}$, $\mathrm{Pr} = \mu_{ref}c_{p,ref}/k_{ref}$, and $\mathrm{Pe} = \mathrm{Re}\,\mathrm{Pr}$ and doing a bit of tedious algebra results in:

$$
\begin{aligned}
&\mathsf{L}(T^*) = \frac{\partial T^*}{\partial t^*} + u^*\frac{\partial T^*}{\partial x^*} - \frac{1}{\mathrm{Pe}}\frac{k^*}{\rho^* c_p^*}\frac{\partial}{\partial x^*}\left(\frac{\partial T^*}{\partial x^*}\right) - \frac{s^*}{\rho^* c_p^*} = 0 && on\ \Omega\\
&\mathsf{l}(T^*(x^* = 0, t^*)) = k^*\frac{\partial T^*}{\partial x^*} + \mathrm{Nu}(T^* - T_r^*) = 0 && on\ \partial\Omega_1\\
&T^*(x^* = 1, t^*) = T_R^* && on\ \partial\Omega_2\\
&T^*(x_0^*, t_o^*) = T_o^*(x^*) && on\ \Omega, \partial\Omega
\end{aligned}
\tag{3}
$$

For constant dimensional values of $u = U$ (convection velocity), $k = k_{ref}$ (thermal conductivity), $\rho = \rho_{ref}$ (density), and $c_p = c_{p,ref}$ (specific heat), the non-dimensional values become $u^* = k^* = \rho^* = c_p^* = 1$. For simplification, we shall assume there to be no source term, hence $s^* = 0$. Finally, we impose pure Dirichlet boundary conditions of $T_{\min}^*$ and $T_{\max}^*$ on $\partial\Omega_1$ and $\partial\Omega_2$ respectively, hence

$$
\begin{aligned}
&\mathsf{L}(q) = \frac{\partial q}{\partial t} + \frac{\partial q}{\partial x} - \frac{1}{Pe}\frac{\partial^2 q}{\partial x^2} = 0 && on\ \Omega\\
&q(x = 0, t) = 0 && on\ \partial\Omega_1\\
&q(x = 1, t) = 1 && on\ \partial\Omega_2\\
&q(x, t_o) = q_o(x) && on\ \Omega, \partial\Omega
\end{aligned}
\tag{4}
$$

where the asterisks have been dropped for convenience and T has been replaced with a generic q.

The above problem statement is termed *the Peclet problem* and is a standard verification problem in the FEA/CFD community as the *steady state solution* is

$$q(x) = \frac{1 - e^{Pe\,x}}{1 - e^{Pe}} \tag{5}$$

It is not uncommon to obtain steady state solutions by “guessing” an initial condition (which may not satisfy the governing PDE!) and then using $\Theta = 1$ to move forward in time until steady state is achieved. Note that, for this problem, the steady state solution is independent of the imposed initial condition - this may not be true for other problem statements!

Homework: For the Peclet problem, verify the non-dimensionalization, obtain the GWS^h, ΘTS, and NIA. Identify the terms which go into {FQ} and [JAC] and their associated Matlab syntax. While we seek the steady state solution, include the transient term in your analysis.