Accuracy and Convergence - Transient Problems

We shall accept as truth that the error in the approximate solution, under uniform spatial and temporal discretization refinement, will behave according to

$$e^h = C_k \Delta x^{2k} + C_t \Delta t \tag{1}$$

which simplifies to our earlier convergence theory for uniform mesh refinement in the steady state. Thus the error consists of two components - a contribution due to the spatial approximation and a contribution due to the temporal approximation. Recalling the convergence rate for uniform mesh refinement to be 2k, we see that the temporal component will converge at a much slower rate. The obvious question becomes "How do we know when our solution has converged?"

The error components can be segregated by first performing a spatial refinement with constant Δt. This holds the temporal error contribution constant and effectively removes it from the analysis. Spatial refinement is then performed either until roundoff error is evident or a suitable convergence criteria is met.

The time step is then refined and the spatial refinement is repeated again. Because the temporal error has been reduced, typically an additional mesh refinement is possible before roundoff is evident.

From the collected data, the convergence rates can obtained and verified.

	t^h			$t^{h/2}$			$t^{h/4}$		
Mesh	Q^h	$e^{h/2}$	$m^{h/4}$	Q^h	$e^{h/2}$	$m^{h/4}$	Q^h	$e^{h/2}$	$m^{h/4}$
Ω^h									
$\Omega^{h/2}$									
$\Omega^{h/4}$									
$\Omega^{h/8}$									
$\Omega^{h/16}$									
$\Omega^{h/32}$									
$\Omega^{h/64}$									

	Ω^h			$\Omega^{h/2}$			$\Omega^{h/4}$		
Mesh	Q^h	$e^{h/2}$	$m^{h/4}$	Q^h	$e^{h/2}$	$m^{h/4}$	Q^h	$e^{h/2}$	$m^{h/4}$
t^h									
$t^{h/2}$									
$t^{h/4}$									
$t^{h/8}$									
$t^{h/16}$									
$t^{h/32}$									
$t^{h/64}$									

Homework:
1. For constant spatial discretization, what is the theoretical $\log_{10}$, $\log_{10}$ convergence rate under uniform temporal refinement?
2. How should the temporal discretization be refined to "keep up" with the uniform spatial m=2 convergence rate?