Transient Heat Conduction

Consider a slab of material experiencing transient heat conduction :

$$T_L = 273\,K$$
$$T_R = 273\,K$$
$$x_L = 0$$
$$x_R = 1\,m$$
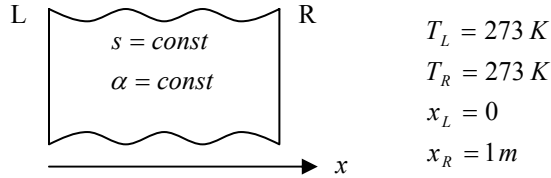
The new additional to this problem is temporal (time) dependence and as such the solution T becomes a function of both t and x. For the following analyses, the unknown solution T will be called Q for sake of generality.

$$\mathcal{L}(Q) = \frac{\partial Q(x,t)}{\partial t} - \alpha \frac{\partial^2 Q(x,t)}{\partial x^2} - s = 0 \qquad \Omega \in (x_L, x_R) \cup t \tag{1}$$

with boundary and initial conditions of

$$\begin{aligned}
Q(x_L, t) &= Q_L & \partial \Omega_1 &\in x_L \\
Q(x_R, t) &= Q_R & \partial \Omega_2 &\in x_R \\
Q(x, to) &= g(x) & \Omega &\in (x_L, x_R)
\end{aligned} \tag{2}$$

Thus, equation (1) has two Dirichlet boundary conditions and an initial condition and the problem statement is therefore well-posed. We begin, as usual, by assuming a continuum approximate form for the unknown solution of temperature Q. Having invested significant theoretical development into $\Psi_\alpha(x)$, we shall opt to put the time variable into the expansion coefficient Q_α, hence

$$Q(x,t) \approx Q^N(x,t) = \sum_{\alpha=1}^{N} \Psi_\alpha(x) Q(t)_\alpha \tag{3}$$

Pushing the approximate form of the solution through the GWS recipe results in

$$GWS^h = S_e \left(\int_{\Omega_e} \{N_k\}\{N_k\}^T \, d\overline{x} \frac{d\{Q\}_e}{dt} + \alpha \int_{\Omega_e} \frac{d\{N_k\}}{dx} \frac{d\{N_k\}^T}{dx} \, d\overline{x} \{Q\}_e - \int_{\Omega_e} \{N_k\}\{N_k\}^T \, d\overline{x} \{S\}_e = \{0\}_e \right) \tag{4}$$

which, upon assembly, can be expressed as the syntactical matrix statement

$$\boxed{[MASS]\frac{d\{Q\}}{dt} + [DIFFA]\{Q\} - \{SRCS\} = \{0\}} \tag{5}$$

where we now have a first order ODE *in time* along with our usual spatially discrete form. Our solution plan now becomes

1) Solve the spatial discretization at the current time level

2) "Somehow" propagate the solution forward some time increment Δt

Let's now figure out how to propagate a scalar ODE equation forward in time and then apply our results to (5).

Department of Mechanical Engineering

Taylor series

Suppose we have a scalar function $f(t)$. Plotting it below

$$f(t) \equiv f(t_n)$$
$$f(t + \Delta t) \equiv f(t_{n+1})$$
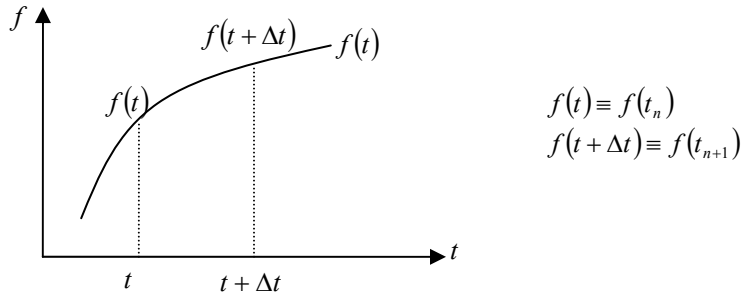
We can use the Taylor Series (TS) to estimate values of the function at specific points via the derivatives.
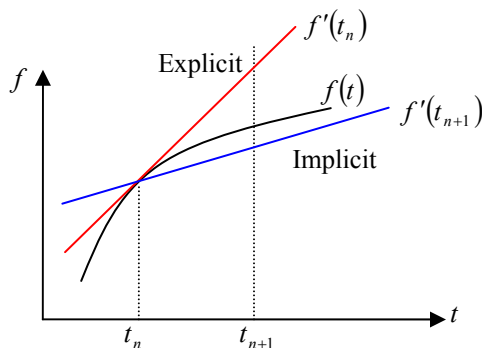
Forward TS :
$$f(t_{n+1}) = f(t_n) + \frac{\Delta t}{1!} f'(t_n) + \frac{\Delta t^2}{2!} f''(t_n) + \frac{\Delta t^3}{3!} f'''(t_n) + \cdots \qquad (6a)$$

Backward TS :
$$f(t_n) = f(t_{n+1}) - \frac{\Delta t}{1!} f'(t_{n+1}) + \frac{\Delta t^2}{2!} f''(t_{n+1}) - \frac{\Delta t^3}{3!} f'''(t_{n+1}) + \cdots \qquad (6b)$$

We can use both (6a) and (6b) to obtain the "new" values of $f(t_{n+1})$ knowing the "old" value of $f(t_n)$ and the derivative of f evaluated at either t_n or t_{n+1}. Note that (6a,b) are infinite series and will need to be truncated to do any numerical work, hence we will be *estimating* the new value of $f(t_{n+1})$. Expressing (6a,b) compactly

Forward TS (Explicit Euler) :
$$f(t_{n+1}) = f(t_n) + \Delta t\, f'(t_n) + O(\Delta t^2) \qquad (6a)$$

Backward TS (Implicit Euler) :
$$f(t_{n+1}) = f(t_n) + \Delta t\, f'(t_{n+1}) + O(\Delta t^2) \qquad (6b)$$

What if we averaged the forward and backward TS?

Trapezoid (Crank-Nicholson) :
$$f(t_{n+1}) = f(t_n) + \frac{\Delta t}{2}\left(f'(t_{n+1}) + f'(t_n)\right) + O(\Delta t^3) \qquad (6c)$$

Department of Mechanical Engineering

We can express all three schemes (6a-c) through an implicitness factor Θ yielding the *Theta Taylor Series* (Θ TS)

$$f(t_{n+1}) = f(t_n) + \Delta t(\Theta f'(t_{n+1}) + (1 - \Theta f'(t_n))) + O(\Delta t^2, \Delta t^3) \qquad (7)$$

where

$$\Theta = 0 \qquad \text{forward TS / Explicit Euler}$$
$$\Theta = 1 \qquad \text{backward TS / Implicit Euler}$$
$$\Theta = \frac{1}{2} \qquad \text{trapezoid rule / Crank - Nicholson}$$

Comments :

Explicit   no iteration required
incredibly small time steps

Implicit   iteration
huge time steps
artificial dissipation (not time accurate, valid for steady state only)

Crank Nicholson   iteration
big time steps
no artificial dissipation (time accurate)

Returning to (5), we have a matrix ODE of the form

$$GWS^h = [MASS]\{Q\}' + [DIFFA]\{Q\} - \{SRCS\} = \{0\} \qquad (8)$$

Applying the Theta Taylor Series (7)

$$\Theta TS = \{Q\}_{n+1} = \{Q\}_n + \Delta t(\Theta\{Q\}'_{n+1} + (1 - \Theta)\{Q\}'_n) + O(\Delta t^2, \Delta t^3) \qquad (9)$$

We must now figure out how to substitute (9) into (8) and then solve the resulting matrix statement.

Transient Heat Conduction   *Page 3 of 3*