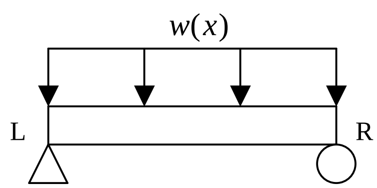
Timeshenko Beam

Consider a simply supported beam with a uniformly distributed load *w(x)* :



$$\left(\frac{dr}{dx}\right)_L = \left(\frac{dr}{dx}\right)_R = 0$$

$$y_L = y_R = 0$$

E = Young's Modulus

G = Shear Modulus

I = Moment of Inertia

A = Cross-Sectional Area

k = Shape Factor

For small deflections, it can be modeled with Timeshenko Beam Theory (Timeshenko, S.P *Theory of Elasticity 3rd Edition* McGraw Hill, 1970). This form was developed to include shear effects into the deflection and is hence more accurate than the Euler-Bernoulli beam which only admits deflection due to moment. Additionally, it precludes the fourth order (biharmonic) Euler-Ernoulli beam equation and creates a linear coupled, two variable system in cross-sectional rotation (*r*) and vertical displacement (*y*)

$$\mathcal{L}(r) = -EI\,\frac{d^2 r(x)}{dx^2} + kGA\left(\frac{d\,y(x)}{dx} + r(x)\right) = 0 \qquad \Omega \in (L, R) \tag{1a}$$

$$\mathcal{L}(y) = -kGA\left(\frac{d^2 y(x)}{dx^2} + \frac{d\,r(x)}{dx}\right) - w(x) = 0 \qquad \Omega \in (L, R) \tag{1b}$$

with boundary conditions of

$$\ell(r) = \frac{dr}{dx} = 0 \qquad \partial\Omega \in L, R \tag{2a}$$

$$y = 0 \qquad \partial\Omega \in L, R \tag{2b}$$

Thus, equation (1a) has two Neumann boundary conditions and equation (1b) has two Dirichlet boundary conditions and the problem statement is therefore well-posed. Note that, unlike the moment/deflection two equation system for the Euler-Bernoulli beam, *both unknowns appear in both equations*. For the ensuing derivations, we shall assume *E*, *I*, *G*, *A*, and *k* to be constant. We begin, as usual, by assuming a continuous, series expansion, approximate form for the unknown solutions of rotation *r* and deflection *y*, hence

$$\begin{aligned} r(x) \approx r^N(x) = \sum_{a=1}^{N} \Psi_a(x) R_a = \Psi_a(x) R_a \quad \text{for} \quad 1 \le a \le N \\ y(x) \approx y^N(x) = \sum_{a=1}^{N} \Psi_a(x) Y_a = \Psi_a(x) Y_a \quad \text{for} \quad 1 \le a \le N \end{aligned} \tag{3}$$

The error is measured by writing the differential operator on the approximation and then weakly enforced to equal zero with the weak statement

$$\begin{aligned} WS_r^N = \int_\Omega \Phi_b(x) \mathcal{L}(r^N) dx = 0 \qquad 1 \le b \le N \\ WS_y^N = \int_\Omega \Phi_b(x) \mathcal{L}(y^N) dx = 0 \qquad 1 \le b \le N \end{aligned} \tag{4}$$

The error in the resulting approximation is minimized via the Galerkin Criterion

$$
\begin{aligned}
GWS_r^N &= \int_\Omega \Psi_b(x)\,\mathsf{L}(r^N)\,dx = 0 \qquad \text{for} \quad 1 \le b \le N \\
GWS_y^N &= \int_\Omega \Psi_b(x)\,\mathsf{L}(y^N)\,dx = 0 \qquad \text{for} \quad 1 \le b \le N
\end{aligned} \tag{5}
$$

Applying the operator

$$
\begin{aligned}
GWS_r^N &= \int_\Omega \Psi_b(x)\left(-EI\,\frac{d^2 r^N(x)}{dx^2} + kGA\left(\frac{d\,y^N(x)}{dx} + r^N(x)\right)\right)dx = 0 \qquad \text{for} \quad 1 \le b \le N \\
GWS_y^N &= \int_\Omega \Psi_b(x)\left(-kGA\left(\frac{d^2 y^N(x)}{dx^2} + \frac{d\,r^N(x)}{dx}\right) - w(x)\right)dx = 0 \qquad \text{for} \quad 1 \le b \le N
\end{aligned} \tag{6}
$$

Integrating by parts and expanding

$$
\begin{aligned}
GWS_r^N &= EI\int_\Omega \frac{d\Psi_b}{dx}\frac{dr^N}{dx}\,dx + kGA\int_\Omega \Psi_b\,\frac{dy^N}{dx}\,dx + kGA\int_\Omega \Psi_b\,r^N\,dx - \Psi_b\,EI\,\frac{dr^N}{dx}\bigg|_{\partial\Omega} = 0 \\
GWS_y^N &= kGA\int_\Omega \frac{d\Psi_b}{dx}\frac{dy^N}{dx}\,dx - kGA\int_\Omega \Psi_b\,\frac{dr^N}{dx}\,dx - \int_\Omega \Psi_b\,w(x)dx - kGA\,\frac{dy^N}{dx}\bigg|_{\partial\Omega} = 0
\end{aligned} \tag{7}
$$

Substituting the series expansion

$$
\begin{aligned}
GWS_r^N &= EI\int_\Omega \frac{d\Psi_b}{dx}\frac{d\Psi_a}{dx}\,dxR_a + kGA\int_\Omega \Psi_b\,\frac{d\Psi_a}{dx}\,dxY_a + kGA\int_\Omega \Psi_b\,\Psi_a\,dxR_a - \Psi_b\,EI\,\frac{dr^N}{dx}\bigg|_{\partial\Omega} = 0 \\
GWS_y^N &= kGA\int_\Omega \frac{d\Psi_b}{dx}\frac{d\Psi_a}{dx}\,dxY_a - kGA\int_\Omega \Psi_b\,\frac{d\Psi_a}{dx}\,dxR_a - \int_\Omega \Psi_b\,w(x)dx - kGA\,\frac{dy^N}{dx}\bigg|_{\partial\Omega} = 0
\end{aligned} \tag{8}
$$

Discretizing

$$
\begin{aligned}
GWS_r^h = \mathsf{S}_e\Bigg(& EI\int_{\Omega_e} \frac{d\{N_k\}}{dx}\frac{d\{N_k\}^T}{dx}\,d\bar{x}\,\{R\}_e + kGA\int_{\Omega_e} \{N_k\}\frac{d\{N_k\}^T}{dx}\,d\bar{x}\{Y\}_e \\
&+ kGA\int_{\Omega_e} \{N_k\}\{N_k\}^T\,d\bar{x}\{R\}_e - \{N_k\}EI\,\frac{dr^N}{dx}\bigg|_{\partial\Omega_e} = \{0\}\Bigg) \\
GWS_y^h = \mathsf{S}_e\Bigg(& kGA\int_{\Omega_e} \frac{d\{N_k\}}{dx}\frac{d\{N_k\}^T}{dx}\,d\bar{x}\,\{Y\}_e - kGA\int_{\Omega_e} \{N_k\}\frac{d\{N_k\}^T}{dx}\,d\bar{x}\{R\}_e \\
&- \int_{\Omega_e} \{N_k\}\{N_k\}^T\,d\bar{x}\{W\}_e - kGA\,\frac{dy^N}{dx}\bigg|_{\partial\Omega_e} = \{0\}\Bigg)
\end{aligned} \tag{9}
$$

Per usual, we can interpolate the distributed load *w(x)* over each element via $w(x)_e \approx \{N_k\}^T\{W\}_e$

Now, can we apply some insight to the flux boundary conditions before we define our terms in (9)?

1. For the rotation equation, the imposed boundary condition is $dr/dx|_{\partial\Omega} = 0$, hence we will be simply adding zero to the right hand side. There is therefore no need to code up the addition of zero. This leads to an important generalization: **the homogenous Neumann boundary condition is free!**
2. For the deflection equation, the imposed boundary condition is $y|_{\partial\Omega} = 0$ and our numeric implementation is to replace the associated entries on the right hand side with the applied Dirichlet data. This leads to another important generalization that we have seen before: **we don't have to code up unknown boundary flux information where we have applied Dirichlet data!**

We now have a matrix statement substantially more complicated than [LHS]*{Q}={RHS}. Identifying the terms

$$GWS_r^h = \mathrm{S}_e\left(EI\int_{\Omega_e} \frac{d\{N_k\}}{dx}\frac{d\{N_k\}^T}{dx}d\overline{x}\,\{R\}_e + kGA\int_{\Omega_e}\{N_k\}\frac{d\{N_k\}^T}{dx}d\overline{x}\{Y\}_e + kGA\int_{\Omega_e}\{N_k\}\{N_k\}^T d\overline{x}\{R\}_e = \{0\}\right)$$

$$[\mathrm{DIFFR}]_e \equiv EI\int_{\Omega_e}\frac{d\{N_k\}}{dx}\frac{d\{N_k\}^T}{dx}d\overline{x} = (EI)(\ )_e\{\ \}_e^T(-1)[\mathrm{A211k}]\{\ \}_e \tag{10a}$$

$$[\mathrm{CONVY}]_e \equiv kGA\int_{\Omega_e}\{N_k\}\frac{d\{N_k\}^T}{dx}d\overline{x} = (kGA)(\ )_e\{\ \}_e^T(\ )[\mathrm{A201k}]\{\ \}_e \tag{10b}$$

$$[\mathrm{MASSR}]_e \equiv kGA\int_{\Omega_e}\{N_k\}\{N_k\}^T d\overline{x} = (kGA)(\ )_e\{\ \}_e^T(1)[\mathrm{A200k}]\{\ \}_e \tag{10c}$$

$$GWS_y^h = \mathrm{S}_e\left(kGA\int_{\Omega_e}\frac{d\{N_k\}}{dx}\frac{d\{N_k\}^T}{dx}d\overline{x}\,\{Y\}_e - kGA\int_{\Omega_e}\{N_k\}\frac{d\{N_k\}^T}{dx}d\overline{x}\{R\}_e = \int_{\Omega_e}\{N_k\}\{N_k\}^T d\overline{x}\{W\}_e\right)$$

$$[\mathrm{DIFFY}]_e \equiv kGA\int_{\Omega_e}\frac{d\{N_k\}}{dx}\frac{d\{N_k\}^T}{dx}d\overline{x} = (kGA)(\ )_e\{\ \}_e^T(-1)[\mathrm{A211k}]\{\ \}_e \tag{10d}$$

$$[\mathrm{CONVR}]_e \equiv kGA\int_{\Omega_e}\{N_k\}\frac{d\{N_k\}^T}{dx}d\overline{x} = (-kGA)(\ )_e\{\ \}_e^T(\ )[\mathrm{A201k}]\{\ \}_e \tag{10e}$$

$$\{\mathrm{LOAD}\}_e \equiv \int_{\Omega_e}\{N_k\}\{N_k\}^T d\overline{x}\{W\}_e = (\ )(\ )_e\{\ \}_e^T(1)[\mathrm{A200k}]\{W\}_e \tag{10f}$$

Upon assembly, (9) can be simplified to the following syntactical matrix statement:

$$\begin{array}{lclclcl} [\text{DIFFR}]\{R\} & + & [\text{MASSR}]\{R\} & + & [\text{CONVY}]\{Y\} & = & \{0\} \\ [\text{CONVR}]\{R\} & & & + & [\text{DIFFY}]\{Y\} & = & \{\text{LOAD}\} \end{array} \tag{11}$$

which can be expressed compactly as

$$\left[\begin{array}{c:c} \text{DIFFR}+\text{MASSR} & \text{CONVY} \\ \hline \text{CONVR} & \text{DIFFY} \end{array}\right]\left\{\begin{array}{c} R \\ \hdashline Y \end{array}\right\} = \left\{\begin{array}{c} 0 \\ \hdashline \text{LOAD} \end{array}\right\} \tag{12}$$

Aha! We have now recovered the familiar matrix statement [LHS]*{Q}={RHS} and are ready to solve! Note that this matrix statement is a bit "bigger" than what we have previously: [LHS] is a 2*nnodes x 2*nnodes square matrix, the column of unknowns {Q} is 2*nnodes x 1, and the {RHS} is 2*nnodes x 1. *Heed the placement of the Dirichlet deflection boundary conditions!*

Homework:

Redo the analysis taking into account a variable mass moment of inertia I(x) and cross-sectional area A(x)